The implicit function theorem and free algebraic sets *

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Abstract: We prove an implicit function theorem for non-commutative functions. We use this to show that if $p(X,Y)$ is a generic non-commuting polynomial in two variables, and X is a generic matrix, then all solutions Y of $p(X,Y)=0$ will commute with X.

1 Introduction

A free polynomial, or nc polynomial (nc stands for non-commutative), is a polynomial in non-commuting variables. Let $\mathbb{P}^d$ denote the algebra of free polynomials in d variables. If $p \in \mathbb{P}^d$, it makes sense to think of p as a function that can be evaluated on matrices. Let $\mathbb{M}_n$ be the set of n-by-n complex matrices, and $\mathbb{M}^{[d]} = \cup_{n=1}^{\infty} \mathbb{M}_n^d$. A free algebraic set is a subset of $\mathbb{M}^{[d]}$ that is the common zero set of a collection of free polynomials.

One principal result in this paper is that, in some generic sense, if X and Y are in $\mathbb{M}_n$ and $p(X,Y)=0$ for some $p \in \mathbb{P}^2$, then Y commutes with X. To explain what we mean by "generically", consider the following specific example. Let a,b,c be complex numbers, and let

$$p(X,Y) \;=\; aX^2 + bXY + cYX.$$

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Then we show in Proposition 9.6 that if $p(X, Y) = 0$, then Y must commute with X unless bX and $-cX$ have a common eigenvalue. We extend this to a general theorem about free algebraic sets defined by $d-1$ polynomials in d variables in Theorem 9.7.

An nc function is a generalization of a free polynomial, just as a holomorphic function in scalar variables can be thought of as a generalization of a polynomial in commuting variables.

To make this precise, define a *graded function* to be a function f, with domain some subset of $\mathbb{M}^{[d]}$, and with the property that if $x \in \mathbb{M}_n^d$, then $f(x) \in \mathbb{M}_n$.

Definition 1.1. An *nc-function* is a graded function f defined on a set $\Omega \subseteq \mathbb{M}^{[d]}$ such that

i) If $x, y, x \oplus y \in \Omega$, then $f(x \oplus y) = f(x) \oplus f(y)$.

ii) If $s \in \mathbb{M}_n$ is invertible and $x, s^{-1}xs \in \Omega \cap \mathbb{M}_n^d$, then $f(s^{-1}xs) = s^{-1}f(x)s$.

Free polynomials are examples of nc-functions. Nc-functions have been studied for a variety of reasons: by Anderson [4] as a generalization of the Weyl calculus; by Taylor [21], in the context of the functional calculus for non-commuting operators; Popescu [15, 16, 17, 18], in the context of extending classical function theory to d-tuples of bounded operators; Ball, Groenewald and Malakorn [5], in the context of extending realization formulas from functions of commuting operators to functions of non-commuting operators; Alpay and Kalyuzhnyi-Verbovetzkii [3] in the context of realization formulas for rational functions that are J-unitary on the boundary of the domain; Helton [7] in proving positive matrix-valued functions are sums of squares; and Helton, Klep and McCullough [8, 9] and Helton and McCullough [10] in the context of developing a descriptive theory of the domains on which LMI and semi-definite programming apply. Recently, Kaliuzhnyi-Verbovetskyi and Vinnikov have written a monograph on the subject [11].

We need to introduce topologies on $\mathbb{M}^{[d]}$. First, we define the *disjoint union topology* by saying that a set U is open in the disjoint union topology if and only if $U \cap \mathbb{M}_n^d$ is open for every n. We shall abbreviate disjoint union as d.u. A set $V \subset \mathbb{M}^{[d]}$ is *bounded* if there exists a positive real number B such that $\|x\| \leq B$ for every x in V.

We shall say that a set $\Omega \subseteq \mathbb{M}^{[d]}$ is an *nc domain* if it is closed under direct sums and unitary conjugations, and is open in the d.u. topology. We

shall say that a topology is an *admissible topology* if it has a basis of bounded nc domains.

Definition 1.2. Let τ be an admissible topology on $\mathbb{M}^{[d]}$, and let Ω be a τ-open set. A τ-holomorphic function is an nc-function $f : \Omega \to \mathbb{M}$ that is τ locally bounded.

Note that if f is a τ-holomorphic function, then for every $a \in \Omega \cap \mathbb{M}_n^d$ and every $h \in \mathbb{M}_n^d$, the derivative

$$Df(a)[h] \;:=\; \lim_{t\to 0} \frac{1}{t}[f(a+th) - f(a)] \tag{1.3}$$

exists [2].

In Section 3 we shall define some particular admissible topologies: the fine, fat, and free topologies. The properties of nc holomorphic functions turn out to depend critically on the choice of topology. In the free topology there is an Oka-Weil theorem, and in particular every free holomorphic function f has the property that $f(x)$ is in the algebra generated by x for every x in the domain [2]; this property was crucial in the authors' study of Pick interpolation for free holomorphic functions [1]. Pointwise approximation of holomorphic functions by polynomials fails for the fine and fat topologies: the following result is a consequence of Theorem 7.7.

Theorem 1.4. For $d \geq 2$, there is a fat holomorphic function that is not pointwise approximable by free polynomials.

The fine and fat topologies do have good properties, though. J. Pascoe proved an Inverse Function theorem for fine holomorphic maps [13]. We extend this in Theorem 5.6 to the fat category. In Theorem 6.1, we prove an Implicit Function theorem in the fine and fat topologies. Here is a special case, when the zero set is of a single function.

Theorem 1.5. Let U an nc domain. Let f be a fine (resp. fat) holomorphic function on U. Suppose that

$$\forall\, a \in U,\ \left[\frac{\partial f}{\partial x^d}(a)[h] = 0\right] \;\Rightarrow h = 0.$$

Let W be the projection onto the first $d-1$ coordinates of $Z_f \cap U$. Then there is a fine (resp. fat) holomorphic function g on W such that

$$Z_f \cap U \;=\; \{(y, g(y)) \,:\, y \in W\}.$$

The advantage of working with the fat topology is that we prove in Theorem 5.5 that if the derivative of a fat holomorphic function is full rank at a point, then it is full rank in a fat neighborhood of the point. This fact, along with the Implicit function theorem, is used to prove Theorem 1.4.

Our final result is that there is no Goldilocks topology. In Theorem 8.6 we show that if τ is an admissible topology on $\mathbb{M}^{[d]}$ with the properties that:

(i) free polynomials are continuous from $(\mathbb{M}^{[d]}, \tau)$ to $(\mathbb{M}^{[1]}, d.u.)$

(ii) τ-holomorphic functions are pointwise approximable by nc polynomials,

then there is no τ Implicit function theorem.

2 Background material

The following lemma is in [9] and [11].

Lemma 2.1. (cf. Lemma 2.6 in [9]). Let Ω be an nc set in $\mathbb{M}^d$, and let f be an nc-function on Ω. Fix $n \geq 1$ and $\Gamma \in \mathbb{M}_n$. If $a, b \in \Omega \cap \mathbb{M}_n^d$ and

$$\begin{bmatrix} b & b\Gamma - \Gamma a \\ 0 & a \end{bmatrix} \in \Omega \cap \mathbb{M}_{2n}^d,$$

then

$$f(\begin{bmatrix} b & b\Gamma - \Gamma a \\ 0 & a \end{bmatrix}) \;=\; \begin{bmatrix} f(b) & f(b)\Gamma - \Gamma f(a) \\ 0 & f(a) \end{bmatrix}. \tag{2.2}$$

If we let $b = a + th$ and $\Gamma = \frac{1}{t}$, and let t tend to 0, we get

Lemma 2.3. Let $U \subseteq \mathbb{M}^{[d]}$ be d.u. open, and suppose that $a \in U$ and $\begin{bmatrix} a & h \\ 0 & a \end{bmatrix} \in U$. Then

$$f(\begin{bmatrix} a & h \\ 0 & a \end{bmatrix}) \;=\; \begin{bmatrix} f(a) & Df(a)[h] \\ 0 & f(a) \end{bmatrix}. \tag{2.4}$$

Combining these two results, we get

Lemma 2.5. Let Ω be an nc domain in $\mathbb{M}^d$, let f be an nc-function on Ω, and let $a \in \Omega$. Then

$$Df(a)[a\Gamma - \Gamma a] \;=\; f(a)\Gamma - \Gamma f(a).$$

By an $\mathcal{L}(\mathbb{C}^\ell, \mathbb{C}^k)$ valued nc function we mean a k-by-ℓ valued matrix of nc functions. An $\mathcal{L}(\mathbb{C}, \mathbb{C}^k)$ valued nc function f can be thought of as a vector of k nc functions, $(f_1, \ldots, f_k)^t$. When $k = d$, we shall call a d-tuple of nc functions on a set in $\mathbb{M}^{[d]}$ an *nc map.*

If Φ is an $\mathcal{L}(\mathbb{C}^\ell, \mathbb{C}^k)$ valued nc function, then if $a \in \mathbb{M}_n^d$, the derivative $D\Phi(a)$ is in $\mathcal{L}(\mathbb{M}_n^d, \mathbb{M}_n \otimes \mathcal{L}(\mathbb{C}^\ell, \mathbb{C}^k))$.

3 Admissible Topologies

3.1 The fine topology

The *fine topology* is the topology that has as a basis all nc domains. Since this is the largest admissible topology, for any admissible topology τ, any τ-holomorphic function is automatically fine holomorphic.

Lemma 3.1. Suppose Ω is an nc domain, and $f : \Omega \to \mathbb{M}$ is d.u. locally bounded. Then f is a fine holomorphic function.

Proof. Let $a \in \Omega$, and $\|f(a)\| = M$. Let $U = \{x \in \Omega : \|f(x)\| < M + 1\}$. Then U is an nc set, and by [2] it is d.u. open. Therefore it is a fine open set. □

It follows from the lemma that the class of nc functions considered in [9, 13] is what we are calling fine holomorphic functions.

J. Pascoe proved the following inverse function theorem in [12]. The equivalence of (i) and (iii) is due to Helton, Klep and McCullough [9].

Theorem 3.2. Let $\Omega \subseteq \mathbb{M}^{[d]}$ be an nc domain. Let Φ be a fine holomorphic map on Ω. Then the following are equivalent:

(i) Φ is injective on Ω.

(ii) $D\Phi(a)$ is non-singular for every $a \in \Omega$.

(iii) The function Φ^{-1} exists and is a fine holomorphic map.

3.2 The Fat topology

Let $\mathbb{R}^+ = \{r \in \mathbb{R} \,|\, r > 0\}$. For $n \in \mathbb{N}$, $a \in \mathbb{M}_n^d$, and $r \in \mathbb{R}^+$, we let $D_n(a, r) \subseteq \mathbb{M}_n^d$ be the matrix polydisc defined by

$$D_n(a, r) = \{x \in \mathbb{M}_n^d \,|\, \max_{1 \le i \le d} \|x_i - a_i\| < r\}. \tag{3.3}$$

If $a \in \mathbb{M}_n^d$, $r \in \mathbb{R}^+$, we define $D(a,r) \subseteq \mathbb{M}^d$ by

$$D(a,r) = \bigcup_{k=1}^{\infty} D_{kn}(a^{(k)}, r), \tag{3.4}$$

where $a^{(k)}$ denotes the direct sum of k copies of a. Finally, if $a \in \mathbb{M}^d$, $r \in \mathbb{R}^+$, we define $F(a,r) \subseteq \mathbb{M}^d$ by

$$F(a,r) = \bigcup_{m=1}^{\infty} \bigcup_{u \in \mathcal{U}_m} u^{-1}\big(D(a,r) \cap \mathbb{M}_m^d\big)\, u, \tag{3.5}$$

where $\mathcal{U}_m$ denotes the set of $m \times m$ unitary matrices.

Lemma 3.6. If $a \in \mathbb{M}^d$ and $r \in \mathbb{R}^+$, then $F(a,r)$ is an nc domain.

Proof. It is immediate from (3.5) that $F(a,r)$ is closed with respect to unitary similarity. To see that $F(a,r)$ is closed with respect to direct sums, assume that $y_1 = u_1^{-1}x_1u_1 \in F(a,r)$ and $y_2 = u_2^{-1}x_2u_2 \in F(a,r)$ where $x_1, x_2 \in D(a,r)$. Noting that (3.3) and (3.4) imply that $x_1 \oplus x_2 \in D(a,r)$ we see that

$$\begin{aligned} y_1 \oplus y_2 &= (u_1^{-1}x_1u_1) \ \oplus \ (u_2^{-1}x_2u_2) \\ &= (u_1 \oplus u_2)^{-1}\big(x_1 \oplus x_2\big)(u_1 \oplus u_2) \\ &\in F(a,r). \end{aligned}$$

□

Lemma 3.7. Let $a, b \in \mathbb{M}^d$, $r, s \in \mathbb{R}^+$ and assume that $x \in F(a,r) \cap F(b,s)$. There exists $\epsilon \in \mathbb{R}^+$ such that $F(x,\epsilon) \subseteq F(a,r) \cap F(b,s)$.

Proof. Choose k, l and u, v so that

$$\|x - u^{-1}a^{(k)}u\| < r \quad \text{and} \quad \|x - v^{-1}b^{(l)}v\| < s$$

and define $\epsilon \in \mathbb{R}^+$ by

$$\epsilon = \min\big\{r - \|x - u^{-1}a^{(k)}u\|,\ s - \|x - v^{-1}b^{(l)}v\|\big\}.$$

We claim that $F(x,\epsilon) \subseteq F(a,r) \cap F(b,s)$. To prove this claim, fix $y \in F(x,\epsilon)$. By the definition of $F(x,\epsilon)$ there exist $m \in \mathbb{N}$ and a unitary w such that

$$\|w^{-1}yw - x^{(m)}\| < \epsilon. \tag{3.8}$$

By the definition of ϵ, $\|x - u^{-1}a^{(k)}u\| \leq r - \epsilon$ so that

$$\|x^{(m)} - (u^{(m)})^{-1}a^{(km)}u^{(m)}\| \leq r - \epsilon. \tag{3.9}$$

As (3.8) and (3.9) imply that $\|w^{-1}yw - (u^{(m)})^{-1}a^{(km)}u^{(m)}\| < r$ which in turn implies that $y \in F(a, r)$. A similar argument implies that $y \in F(b, s)$. □

Lemma 3.7 guarantees that the sets of the form $D(a, r)$ with $a \in \mathbb{M}^d$ and $r \in \mathbb{R}^+$ form a basis for a topology on $\mathbb{M}^d$. We refer to this topology as the *fat* topology.

3.3 The free topology

The third example of an admissible topology is the *free topology.* A *basic free open set* in $\mathbb{M}^{[d]}$ is a set of the form

$$G_\delta \;=\; \{x \in \mathbb{M}^{[d]} : \|\delta(x)\| < 1\},$$

where δ is a J-by-J matrix with entries in $\mathbb{P}^d$. We define the free topology to be the topology on $\mathbb{M}^{[d]}$ which has as a basis all the sets G_δ, as J ranges over the positive integers, and the entries of δ range over all polynomials in $\mathbb{P}^d$. (Notice that $G_{\delta_1} \cap G_{\delta_2} = G_{\delta_1 \oplus \delta_2}$, so these sets do form the basis of a topology). The free topology is a natural topology when considering semi-algebraic sets.

Proposition 3.10. The fat topology is an admissible topology, finer than the free topology and coarser than the fine topology.

Proof. All that needs to be shown is that for any G_δ and any $x \in G_\delta$, there is a fat neighborhood of x in G_δ. But this is obvious, because δ is a finite matrix of free polynomials. □

4 Hessians

Let f be an nc function defined on a d.u. open set $U \subseteq \mathbb{M}^{[d]}$, and let $a \in U$. We define the *Hessian of f at a* to be the bilinear form $Hf(a)$ defined on $\mathbb{M}^d \times \mathbb{M}^d$ by the formula

$$Hf(a)[h, k] = \lim_{t \to 0} \frac{Df(a + tk)[h] - Df(a)[h]}{t}, \qquad h, k \in \mathbb{M}^d.$$

If $A \subseteq \mathbb{M}^d$ and $B \subseteq \mathbb{M}^b$ we define $A\ [\times]\ B \subseteq \mathbb{M}^{d+b}$ by

$$A\ [\times]\ B = \bigcup_{n=1}^{\infty} (A \cap \mathbb{M}_n^d) \times (B \cap \mathbb{M}_n^b).$$

If τ is a topology on $\mathbb{M}^d$ and σ is a topology on $\mathbb{M}^b$, then we let $\tau\ [\times]\ \sigma$ be the topology on $\mathbb{M}^{d+b}$ that has a basis

$$\tau\ [\times]\ \sigma = \bigcup \{A\ [\times]\ B \,|\, A \in \tau, B \in \sigma\}.$$

If τ and σ are admissible, then $\tau\ [\times]\ \sigma$ is admissible.

Lemma 4.1. Let τ be an admissible topology on $\mathbb{M}^d$ and assume that $f : \Omega \to \mathbb{M}^1$ is a τ holomorphic function. If σ is any admissible topology, then g defined on $\Omega\ [\times]\ \mathbb{M}^d$ by the formula

$$g(x,h) = Df(x)[h], \qquad (x,h) \in \Omega\ [\times]\ \mathbb{M}^d,$$

is a $\tau\ [\times]\ \sigma$ holomorphic function. Furthermore, for each fixed $n \in \mathbb{N}$ and $x \in \Omega \cap \mathbb{M}_n^d$, $g(x,h)$ is a bounded linear map from $\mathbb{M}_n^d$ to $\mathbb{M}_n^1$.

Lemma 4.2. Let $\Omega \subseteq \mathbb{M}^d$ be a fine domain, $f : \Omega \to \mathbb{M}^1$ a fine holomorphic function and $a \in \Omega$. If h and k are sufficiently small, then

$$f(\begin{bmatrix} a & k & h & 0 \\ 0 & a & 0 & h \\ 0 & 0 & a & k \\ 0 & 0 & 0 & a \end{bmatrix}) = \begin{bmatrix} f(a) & Df(a)[k] & Df(a)[h] & Hf(a)[h,k] \\ 0 & f(a) & 0 & Df(a)[h] \\ 0 & 0 & f(a) & Df(a)[k] \\ 0 & 0 & 0 & f(a) \end{bmatrix}.$$

Proof. Let

$$X = \begin{bmatrix} a & k \\ 0 & a \end{bmatrix}$$

and

$$H = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}.$$

Define a function $g(x,h)$ by

$$g(x,h) = Df(x)[h], \quad (x,h) \in \Omega\ [\times]\ \mathbb{M}^d.$$

By Lemma 4.1 g is a fine holomorphic function of $2d$ variables. Hence, by Lemma 2.3,

$$\begin{aligned} g(X,H) &= g(\begin{bmatrix} (a,h) & (k,0) \\ 0 & (a,h) \end{bmatrix}) \\ &= \begin{bmatrix} g(a,h) & Dg(a,h)[k,0] \\ 0 & g(a,h) \end{bmatrix}. \end{aligned}$$

But

$$\begin{aligned} Dg(a,h)[k,0] &= \lim_{t\to 0} \frac{g(a+tk,h)-g(a,h)}{t} \\ &= \lim_{t\to 0} \frac{Df(a+tk)[h]-Df(a)[h]}{t} \\ &= Hf(a)[h,k]. \end{aligned}$$

Therefore,

$$g(X,H) = \begin{bmatrix} Df(a)[h] & Hf(a)[h,k] \\ 0 & Df(a)[h] \end{bmatrix}.$$

Using this last formula and Lemma 2.3 several times we have that

$$\begin{aligned} f(\begin{bmatrix} a & k & h & 0 \\ 0 & a & 0 & h \\ 0 & 0 & a & k \\ 0 & 0 & 0 & a \end{bmatrix}) &= f(\begin{bmatrix} X & H \\ 0 & X \end{bmatrix}) \\ &= \begin{bmatrix} f(X) & g(X,H) \\ 0 & f(X) \end{bmatrix} \\ &= \begin{bmatrix} f(a) & Df(a)[k] & Df(a)[h] & Hf(a)[h,k] \\ 0 & f(a) & 0 & Df(a)[h] \\ 0 & 0 & f(a) & Df(a)[k] \\ 0 & 0 & 0 & f(a) \end{bmatrix}. \end{aligned}$$

□

5 Extending non-singularity to a fat neighborhood

Lemma 5.1. Suppose that $f : U \to \mathbb{M}^1$ is a fat holomorphic function. For each $a \in U$, there exists $r \in \mathbb{R}^+$ such that Hf is a uniformly bounded bilinear form on $F(a,r)$.

Proof. Fix $a \in U$. Since f is a fat holomorphic function, there exists $s, \rho \in \mathbb{R}^+$ such that $F(a,s) \subseteq U$, f is a fine holomorphic function on $F(a,s)$, and

$$\sup_{x \in F(a,s)} \|f(x)\| \leq \rho.$$

Let $r = s/2$. If $x \in F(a,r)$, then by the triangle inequality if $\|h\|, \|k\| < r/2$, then

$$\begin{bmatrix} x & k & h & 0 \\ 0 & x & 0 & h \\ 0 & 0 & x & k \\ 0 & 0 & 0 & x \end{bmatrix} \in F(a,s).$$

Hence, by Lemma 4.2,

$$\|Hf(x)[h,k]\| \leq \rho$$

whenever $x \in F(a,r)$ and $\|h\|, \|k\| < r/2$. It follows that if $x \in F(a,r)$, then

$$\|Hf(x)[h,k]\| \leq \frac{r^2\rho}{2}\|h\|\|k\|$$

for all h and k. □

Now, let $\Omega \subseteq \mathbb{M}^d$ be a fine domain, $f : \Omega \to \mathbb{M}^1$ a fine holomorphic function and $a \in \Omega \cap \mathbb{M}_n^d$. We set $L = Df(a)$. If L is nonsingular (i.e. surjective), then for each $k \in \mathbb{N}$, $\mathrm{id}_k \otimes L = Df(a^{(k)})$ is nonsingular as well. Thus, if we set $L_k = \mathrm{id}_k \otimes L$, then for each k, L_k has a right inverse, i.e., a bounded transformation $R : \mathbb{M}_{kn}^1 \to \mathbb{M}_{kn}^d$ such that $L_k R = 1$.

Definition 5.2. Let us agree to say that L *is completely nonsingular* if

$$\sup_k \inf\{\|R\| \mid R \text{ is a right inverse of } L_k\} < \infty.$$

If L is completely nonsingular, we define $c(L)$ by

$$c(L) = \Big(\sup_k \inf\{\|R\| \mid R \text{ is a right inverse of } L_k\}\Big)^{-1}$$

Lemma 5.3. If $L : \mathbb{M}_n^d \to \mathbb{M}_n^1$ is linear and has a right inverse R, then L is completely non-singular and $c(L) \geq 1/(n\|R\|)$.

Proof. Note that $\mathrm{id}_k \otimes R$ is a right inverse of $\mathrm{id}_k \otimes L$. Therefore $c(L)$ is at least the reciprocal of

$$\|R\|_{cb} := \sup_k \|\mathrm{id}_k \otimes R\|.$$

By a result of R. Smith [19]; [14, Prop 8.11], any linear operator T defined on an operator space and with range $\mathbb{M}_n$ has $\|T\|_{cb} = \|\mathrm{id}_n \otimes T\| \leq n\|T\|$. But R is just a d-tuple of linear operators from $\mathbb{M}_n$ to $\mathbb{M}_n$, so $\|R\|_{cb} \leq n\|R\|$. □

Lemma 5.4. If L is completely nonsingular, $k \in \mathbb{N}$, $E : \mathbb{M}_{kn}^d \to \mathbb{M}_{kn}$ is linear, and $\|E\| < c(L)$, then $L_k + E$ is nonsingular.

Proof. Assume that L is completely nonsingular, $k \in \mathbb{N}$, $E : \mathbb{M}_{kn}^d \to \mathbb{M}_{kn}$, and $\|E\| < c(L)$. Choose $R : \mathbb{M}_{kn}^1 \to \mathbb{M}_{kn}^d$ satisfying $L_k R = 1$ and $\|R\| \leq c(L)^{-1}$.

If $\|E\| < c(L)$, then $\|ER\| < 1$ and as a consequence, $1 + ER$ is invertible. But

$$\begin{aligned}(L_k + E)R(1 + ER)^{-1} &= (L_k R + ER)(1 + ER)^{-1} \\ &= (1 + ER)(1 + ER)^{-1} \\ &= 1.\end{aligned}$$

Hence, if $\|E\| < c(L)$, then $L_k + E$ is surjective. □

Theorem 5.5. Let $U \subseteq \mathbb{M}^d$ be a fat nc domain and assume that $f : U \to \mathbb{M}^\ell$ is a fat holomorphic function. Let $a \in U \cap \mathbb{M}_n^d$.

(i) If $Df(a)$ is full rank, then there exists a fat domain Ω such that $a \in \Omega \subseteq U$ and $Df(x)$ is full rank for all $x \in \Omega$.

(ii) If $\ell \leq d$ and $Df(a)$ is an isomorphism from

$$0^{d-\ell} \times \mathbb{M}_n^\ell := \{(0, \ldots, 0, h^{d-\ell+1}, \ldots, h^d) : h^r \in \mathbb{M}_n, d - \ell + 1 \leq r \leq d\}$$

onto $\mathbb{M}_n^\ell$, then there is a fat domain Ω such that $a \in \Omega \subseteq U$ and $Df(x)$ is nonsingular on $0^{d-\ell} \times \mathbb{M}_\mu^\ell$ for all $\mu \in \mathbb{N}$ and for all $x \in \Omega \cap \mathbb{M}_\mu^d$.

Proof. Let $f = (f^1, \ldots, f^\ell)^t$. By Lemma 5.1, there exist $s, M \in \mathbb{R}^+$ such that, for each $1 \leq j \leq \ell$,

$$\|Hf^j(x)[h, k]\| \leq M\|h\|\|k\|$$

for all $x \in F(a,s)$ and all $h,k \in \mathbb{M}^d$ that have the same size as x. Choose $r \in \mathbb{R}^+$ satisfying

$$r < \min\big\{s, \frac{c(Df(a))}{M\sqrt{\ell}}\big\}.$$

Let $m \in \mathbb{N}$ and $x \in F(a,r) \cap \mathbb{M}^d_{mn}$ (so that $\|x - a^{(m)}\| < r$). We have that for each j

$$\begin{aligned}
\|Df^j(x)[h] - Df^j(a^{(m)})[h]\| &= \|\int_0^1 \frac{d}{dt} Df^j\big(a^{(m)} + t(x-a^{(m)})\big)[h]dt\| \\
&= \|\int_0^1 Hf^j\big(a^{(m)} + t(x-a^{(m)})\big)[h, x-a^{(m)}]dt\| \\
&\leq M\|h\|\|x-a^{(m)}\| \\
&< \frac{c(Df(a))}{\sqrt{\ell}}\|h\|.
\end{aligned}$$

So

$$\|Df(x) - Df(a^{(m)})\| < c(Df(a)).$$

Hence, by Lemma 5.4, $Df(x)$ is nonsingular, proving (i).

Part (ii) follows in the same way, by considering $Df(x)|_{0^{d-\ell}\times\mathbb{M}^\ell_m}$. By hypothesis, this has a right inverse at a, so by Lemma 5.3 is completely nonsingular. Therefore there is a fat neighborhood of a (perhaps smaller than in case (i)) on which $Df(x)|_{0^{d-\ell}\times\mathbb{M}^\ell_m}$ is nonsingular. □

We can now prove a fat version of the inverse function theorem, Theorem 3.2.

Theorem 5.6. Let $\Omega \subseteq \mathbb{M}^{[d]}$ be a fat nc domain. Let Φ be a fat holomorphic map on Ω. Then the following are equivalent:

(i) Φ is injective on Ω.

(ii) $D\Phi(a)$ is non-singular for every $a \in \Omega$.

(iii) The function Φ^{-1} exists and is a fat holomorphic map.

Proof. In light of Pascoe's Theorem 3.2, all that remains to prove is that Asumption (ii) implies that Φ^{-1} is fat holomorphic. Let $U = \Phi(\Omega)$, and let $b = \Phi(a) \in U \cap \mathbb{M}^d_n$. We must find a fat neighborhood of b on which Φ^{-1} is bounded. This in turn will follow if we can find $r, s > 0$ such that

$$\Phi(D(a,r)) \supseteq D(b,s), \tag{5.7}$$

where $D(a,r)$ is defined in (3.4). By Lemma 5.1, there exists $r_1 > 0, M$ such that the Hessian of f is bounded by M on $F(a,r_1)$. Choose $0 < r < r_1$ so that

$$Mr \ < \ \frac{1}{2}c(D\Phi(a)),$$

and choose $s > 0$ so that

$$s \ < \ \frac{r}{2}c(D\Phi(a)).$$

We claim that with these choices, (5.7) holds.

Indeed, choose $k \in \mathbb{N}$, and let $x \in D_{kn}(a^{(k)}, r)$. Let us write α for $a^{(k)}$. Then

$$\begin{aligned}
\|\Phi(x) - \Phi(\alpha)\| &= \| \int_0^1 \frac{d}{dt}\Phi(\alpha + t(x-\alpha))dt\| \\
&= \| \int_0^1 D\Phi(\alpha + t(x-\alpha))[x-\alpha]dt\| \\
&= \|D\Phi(\alpha)[x-\alpha] + \\
&\quad \int_0^1 D\Phi(\alpha + t(x-\alpha))[x-\alpha] - D\Phi(\alpha)[x-\alpha]dt\| \\
&\geq \|D\Phi(\alpha)[x-\alpha]\| - M\|x-\alpha\|^2 \\
&\geq (c(D\Phi(a)) - M\|x-\alpha\|)\,\|x-\alpha\| \\
&\geq \frac{1}{2}c(D\Phi(a))\|x-\alpha\|.
\end{aligned}$$

Since $D\Phi$ is non-singular, we have that $\Phi(D_{kn}(a^{(k)}, r))$ is an open connected set, and by the last inequality it contains $D_{kn}(b^{(k)}, s)$. □

6 The implicit function theorem

Let $f = (f_1, \ldots, f_k)$ be an $\mathcal{L}(\mathbb{C}, \mathbb{C}^k)$ valued nc function. We shall let $Z_f = \cap_{i=1}^k Z_{f_i}$ denote the zero set of f. If $a \in \mathbb{M}_n^d$, the derivative of f at a, $Df(a)$, is a linear map from $\mathbb{M}_n^d$ to $\mathbb{M}_n^k$. We shall say that $Df(a)$ is of *full rank* if the rank of this linear map is kn^2.

For convenience in the following theorem, we shall write h in $\mathbb{M}_n^k$ as $h = (h^{d-k+1}, \ldots, h^d)$.

Theorem 6.1. Let U an nc domain. Let f be an $\mathcal{L}(\mathbb{C}, \mathbb{C}^k)$ valued fine holomorphic function on U, for some $1 \leq k \leq d-1$. Suppose

$$\begin{aligned} &\forall\, n \in \mathbb{N},\ \forall\, a \in U \cap \mathbb{M}_n^d, \\ &\quad \forall\, h \in \mathbb{M}_n^k \setminus \{0\}, \quad Df(a)[(0, \ldots, 0, h^{d-k+1}, \ldots, h^d)] \neq 0. \end{aligned} \tag{6.2}$$

Let W be the projection onto the first $d-k$ coordinates of $Z_f \cap U$. Then there is an $\mathcal{L}(\mathbb{C}, \mathbb{C}^k)$-valued fine holomorphic function g on W such that

$$Z_f \cap U \ = \ \{(y, g(y)) \,:\, y \in W\}.$$

Moreover, if f is fat holomorphic, then g can also be taken to be fat holomorphic.

Proof. Let $\Phi(x) = (x^1, \ldots, x^{d-k}, f(x))^t$ be the nc map defined on U by prepending the first $d-k$ coordinate functions. By (6.2), Φ is non-singular on U, so by Theorem 3.2, there is an nc map from U onto some set Ω, with inverse Ψ.

Let us write points x in $\mathbb{M}_n^d$ as (y, z), where $y \in \mathbb{M}_n^{d-k}$ and $z \in \mathbb{M}_n^k$. Then y is in W iff there is some z such that $(y, z) \in U$ and $f(y, z) = 0$.

Let $\Psi = \psi_1 \oplus \psi_2$, where ψ_1 is Ψ followed by projection onto the first $d-k$ coordinates, and ψ_2 is Ψ followed by projection onto the last k coordinates. Define $g(y) = \psi_2(y^1, \ldots, y^{d-k}, 0, \ldots, 0)$.

If $(y, z) \in Z_f \cap U$, then $\Phi(y, z) = (y, 0)$ and

$$\Psi \circ \Phi(y, z) \ = \ (y, z) \ = \ (\psi_1(y, 0), g(y)),$$

so $z = g(y)$.

Conversely, if $y \in W$ and $z = g(y)$, then $\Psi(y, 0) = (\psi_1(y, 0), g(y))$, so

$$\Phi \circ \Psi(y, 0) \ = \ (y, 0) \ = \ (\psi_1(y, 0), f(\psi_1(y, 0), g(y)).$$

Therefore $f(y, g(y)) = 0$.

Finally, if f is fat holomorphic, then by Theorem 5.6 the function Ψ is fat holomorphic, and hence so is g. □

Two questions naturally arise. The first is whether satisfying (6.2) at a particular point automatically leads to it holding on a neighborhood. Theorem 5.5 shows that this is true in the fat category.

The second question is whether whenever $Df(a)$ is of full rank, one can change basis to obtain condition (6.2). We shall show in Corollary 6.15 that the answer generically is yes.

Definition 6.3. Let $d \geq 2$. We shall say that a d-tuple $x \in \mathbb{M}_n^d$ is *broad* if

$$\{p(x) : p \in \mathbb{P}^d\} \;=\; \mathbb{M}_n.$$

Theorem 6.4. Let $d \geq 2$, let $a \in \mathbb{M}_n^d$, and assume a is broad. Let $N \leq (d-1)n^2 + 1$. Suppose $H_1, \ldots, H_N \in \mathbb{M}_n^d$ are linearly independent modulo $\{a\Gamma - \Gamma a \;:\; \Gamma \in \mathbb{M}_n\}$. Then, for every $K_1, \ldots, K_N \in \mathbb{M}_n$ and for every $M \in \mathbb{M}_n$ there exists $p \in \mathbb{P}^d$ such that

$$p(a) \;=\; M, \quad \text{and} \quad Dp(a)[H_i] = K_i, \forall\, i \leq N. \tag{6.5}$$

Proof. We shall prove the theorem by induction on N. When $N = 0$, the conclusion holds because a is broad. So assume that the theorem has been proved for some $0 \leq N \leq (d-1)n^2$, and we wish to show the conclusion holds for $N+1$. Fix $H_1, \ldots, H_{N+1}$. Assume that

$$H_{N+1} \;\notin\; \{a\Gamma - \Gamma a : \Gamma \in \mathbb{M}_n\} + \vee\{H_1, \ldots, H_N\}. \tag{6.6}$$

Let

$$I \;=\; \{p \in \mathbb{P}^d : p(a) = 0, Dp(a)[H_i] = 0, i \leq N\}.$$

Case 1: $N \geq 1$, and for all $p \in I$, we have $Dp(a)[H_{N+1}] = 0$.
If this holds, then by Lemma 2.3 the map

$$\pi : \; p\Bigg(\begin{bmatrix} \begin{bmatrix} a & H_1 \\ 0 & a \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} a & H_N \\ 0 & a \end{bmatrix} \end{bmatrix}\Bigg) \;\mapsto\; p\Bigg(\begin{bmatrix} a & H_{N+1} \\ 0 & a \end{bmatrix}\Bigg)$$

is a well-defined homomorphism, as p ranges over $\mathbb{P}^d$. By the inductive hypothesis and Lemma 2.3, we have that for all $K = (K_1, \ldots, K_N)$,

$$\pi : \; \begin{bmatrix} \begin{bmatrix} M & K_1 \\ 0 & M \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} M & K_N \\ 0 & M \end{bmatrix} \end{bmatrix} \;\mapsto\; \begin{bmatrix} M & L(M,K) \\ 0 & M \end{bmatrix}$$

for some linear map L. Letting $K = 0$ and using the fact that π is multiplicative, we get

$$M_1 L(M_2, 0) + L(M_1, 0) M_2 \;=\; L(M_1 M_2, 0).$$

This means that the map $M \mapsto L(M, 0)$ is a derivation on $\mathbb{M}_n$, so it must be inner [6, Thm 3.22]. Therefore there exists $\Gamma \in \mathbb{M}_n$ such that

$$L(M, 0) \;=\; M\Gamma - \Gamma M. \tag{6.7}$$

As

$$\begin{bmatrix} \begin{bmatrix} M & K_1 \\ 0 & M \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} M & K_N \\ 0 & M \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & K_1 \\ 0 & 0 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} 0 & K_N \\ 0 & 0 \end{bmatrix} \end{bmatrix} \tag{6.8}$$

on the one hand maps to

$$\begin{bmatrix} M & L(M, K) \\ 0 & M \end{bmatrix} \begin{bmatrix} 0 & L(0, K) \\ 0 & 0 \end{bmatrix}$$

and on the other to

$$\begin{bmatrix} 0 & L(0, MK) \\ 0 & 0 \end{bmatrix},$$

we conclude

$$L(0, MK) \;=\; ML(0, K), \tag{6.9}$$

and by reversing the factors in (6.8) get

$$L(0, KM) \;=\; L(0, K)M. \tag{6.10}$$

Let $E_i \in \mathbb{M}_n^N$ have the identity in the i^{th} slot, and 0 elsewhere. By (6.9) and (6.10), we have $L(0, E_i)$ commutes with every matrix in $\mathbb{M}_n$, so must be a scalar. By linearity and (6.9) again, we get that

$$L(0, K) \;=\; \sum_{i=1}^{N} c_i K_i. \tag{6.11}$$

As π is linear, we have $L(M,K) = L(M,0) + L(0,K)$, so combining this observation with (6.7) and (6.11), we conclude that

$$L(M,K) \;=\; M\Gamma - \Gamma M + \sum_{i=1}^{N} c_i K_i. \tag{6.12}$$

By Lemma 2.3, this means

$$Dp(a)[H_{N+1}] \;=\; a\Gamma - \Gamma a + \sum_{i=1}^{N} c_i Dp(a)[H_i]. \tag{6.13}$$

Let $p(x) = x^r$, the r^{th} coordinate function, in (6.13). This yields

$$H^r_{N+1} \;=\; a\Gamma - \Gamma a + \sum_{i=1}^{N} c_i H^r_i. \tag{6.14}$$

As (6.14) holds for $1 \leq r \leq d$ with the same Γ, this contradicts (6.6).

Case 2: $N = 0$, and for all $p \in I$, we have $Dp(a)[H_1] = 0$.

Now the inductive hypothesis is that for all $M \in \mathbb{M}_n$, there is a polynomial p with $p(a) = M$. The ideal I is all polynomials that vanish at a. As in Case 1, we conclude that the map

$$\pi : p(a) \;\to\; p(\begin{bmatrix} a & H_1 \\ 0 & a \end{bmatrix}) \;=\; \begin{bmatrix} p(a) & Dp(a)[H_1] \\ 0 & p(a) \end{bmatrix}$$

is a well-defined homomorphism, and that

$$Dp(a)[H_1]$$

is a derivation on $\{p(a)\}$, so

$$Dp(a)[H_1] \;=\; p(a)\Gamma - \Gamma p(a)$$

for some $\Gamma \in \mathbb{M}_n$. Letting p be each of the coordinate functions in turn, we get $H_1 = a\Gamma - \Gamma a$, a contradiction to (6.6).

Case 3: As the previous two cases have been ruled out, we must be in the situation that for some $p \in I$, $Dp(a)[H_{N+1}] \neq 0$. As

$$Dqp(a)[H] \;=\; Dq(a)[H]p(a) + q(a)Dp(a)[H],$$

we have that

$$\mathcal{D} \;:=\; \{Dp(a)[H_{N+1}] : p \in I\}$$

is invariant under multiplication on the left or right by elements of

$$\{q(a) : q \in I\}.$$

Since a is broad, we have that $\mathcal{D}$ is a non-empty ideal in $\mathbb{M}_n$, and therefore all of $\mathbb{M}_n$.

Choose now M and $K_1, \ldots, K_{N+1}$ in $\mathbb{M}_n$. By the inductive hypothesis, we can find a polynomial q such that

$$q(a) \;=\; M, \quad \text{and} \quad Dq(a)[H_i] = K_i, \forall\, i \leq N.$$

Since we are in Case 3, there is a polynomial $p \in I$ such that

$$Dp(a)[H_{N+1}] \;=\; K_{N+1} - Dq(a)[H_{N+1}].$$

Then the polynomial $r = p + q$ satisfies

$$r(a) \;=\; M, \quad \text{and} \quad Dr(a)[H_i] = K_i, \forall\, i \leq N+1.$$

□

As a consequence, if $Df(a)$ is of full rank, then, generically, there is a polynomial change of variables that allows one to assume it is of full rank on $0^{d-k} \oplus \mathbb{M}_n^k := \{(0, \ldots, 0, h) : h \in \mathbb{M}_n^k\}$.

Corollary 6.15. Let $d \geq 2$, let $\Omega \subseteq \mathbb{M}^{[d]}$ be an nc domain, and fix $1 \leq k \leq d-1$. Let f be an $\mathcal{L}(\mathbb{C}, \mathbb{C}^k)$ valued fine holomorphic function on Ω. Suppose that $Df(a)$ is of rank kn^2 for some point $a \in \Omega \cap \mathbb{M}_n^d$. Suppose also that $(a^1, \ldots, a^{d-k}, f(a))$ is broad, and the commutant of a is $\mathbb{C}$.

Then there are a d.u open set U containing a broad point b and an invertible nc polynomial map Φ from U into Ω, mapping the point b to a, such that,

$$\begin{aligned} &\forall\, h = (h^{d-k+1}, \ldots, h^d) \in \mathbb{M}_n^k \setminus \{0\} \\ &Df \circ \Phi(b)[(0, \ldots, 0, h^{d-k+1}, \ldots, h^d)] \neq 0. \end{aligned} \tag{6.16}$$

Moreover, if Ω is fat, then U can be chosen to be a fat nc domain.

Proof. Choose $b = (a^1, \ldots, a^{d-k}, f(a))$. By the chain rule, (6.16) will hold provided

$$\left\{D\Phi(b)[\{0^{d-k} \oplus \mathbb{M}_n^k\}]\right\} \cap \ker Df(a) = \{0\}. \tag{6.17}$$

By Theorem 6.4 we can choose the polynomial entries $(p^1, \ldots, p^d)$ of Φ so that $\Phi(b) = a$ and the action of the derivative is arbitrary, except on the set $\{b\Gamma - \Gamma b\}$. But on this set, by Lemma 2.5, we have

$$D\Phi(b)[b\Gamma - \Gamma b] = \Phi(b)\Gamma - \Gamma\Phi(b) = a\Gamma - \Gamma a. \tag{6.18}$$

If this were in the kernel of $Df(a)$, we would have

$$0 = Df(a)[a\Gamma - \Gamma a] = f(a)\Gamma - \Gamma f(a).$$

But if this holds, and $b\Gamma - \Gamma b$ is in $\{0^{d-k} \oplus \mathbb{M}_n^k\}$, then $b\Gamma - \Gamma b = 0$.

So for any choice of Φ with $\Phi(b) = a$, we have

$$\left\{D\Phi(b)[\{b\Gamma - \Gamma b\} \cap \{0 \oplus \mathbb{M}_n^k\}]\right\} \cap \ker Df(a) = \{0\}.$$

As b is broad and $\{a\}' = \mathbb{C}$, the sets $\{b\Gamma - \Gamma b\}$ and $\{a\Gamma - \Gamma a\}$ are both of dimension $n^2 - 1$. Now choose the derivatives of Φ in a set of directions that complements $\{b\Gamma - \Gamma b\}$ so that $D\Phi(b)$ is of full rank and (6.17) holds. Let

$$U = \Phi^{-1}(\Omega) \cap \{x : D\Phi(x) \text{ is invertible}\}. \tag{6.19}$$

Finally, if Ω is fat, then choose U to be the intersection of the fat nc domain $\Phi^{-1}(\Omega)$ with a fat neighborhood of b on which $D\Phi$ is invertible, which exists by Theorem 5.5. □

7 The range of an nc function

A necessary and sufficient condition that the function

$$f : s^{-1}xs \mapsto s^{-1}zs$$

is well-defined on the similarity orbit S_x of x is that z be in $\{x\}''$. So if f is an nc function on a d.u. open set, then for every M in the commutant of x, $1 + tM$ must commute with $f(x)$ for t small. This imposes the requirement that

$$f(x) \in \{x\}''.$$

When $d = 1$, we have $\mathcal{A}_x = \{x\}''$, but this containment can be proper for $d > 1$. (By $\mathcal{A}_x$ we mean the algebra generated by x).

Question 7.1. If f is a τ nc function on an τ open set U, is $f(x) \in \mathcal{A}_x$?

A necessary condition for f to be pointwise approximable by polynomials is that $f(x) \in \mathcal{A}_x$. In [2], the authors proved that a free holomorphic function is locally the uniform limit of free polynomials, so the answer to Question 7.1 is yes for the free topology.

We shall show that the answer is no for the fat (and hence for the fine) topology.

Indeed, let $x_0 \in \mathbb{M}_2^2$ be

$$x_0 \;=\; \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right],$$

and let $z_0 \in \mathbb{M}^2$ be

$$z_0 \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

As $\{x_0\}'$ is just the scalars, we have $z_0 \in \{x_0\}'' \setminus \mathcal{A}_{x_0}$, and the function

$$f : s^{-1}x_0 s \;\mapsto\; s^{-1}z_0 s \tag{7.2}$$

is well-defined on the similarity orbit S_{x_0} of x_0. We shall show that it extends to a fat holomorphic function.

Define p by

$$p(X,Y,Z) \;=\; (Z)^2 + XZ + ZX + YZ - \mathrm{id}. \tag{7.3}$$

If $x_0 = (X, Y)$ and $z_0 = Z$ are substituted in (7.3), we get $p(x_0, z_0) = 0$. Let

$$a \;=\; \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right], \tag{7.4}$$

Lemma 7.5.

$$\frac{\partial}{\partial Z} p(a)[h] \;=\; \begin{pmatrix} h_{11} + h_{12} + h_{21} & h_{11} + h_{12} + h_{22} \\ h_{11} + h_{22} & h_{12} + h_{21} \end{pmatrix}. \tag{7.6}$$

It is immediate from (7.6) that $\frac{\partial}{\partial Z} p(a) : \mathbb{M}_2 \to \mathbb{M}_2$ is onto, and so has a right inverse. By Theorem 5.5, there is a fat domain $\Omega \ni a$ such that $\frac{\partial}{\partial Z} p(\lambda)$ is non-singular for all $\lambda \in \Omega$.

Now we invoke Theorem 6.1. Let V be the projection onto the first two coordinates of Ω. This is a fat domain containing x_0. We conclude:

Theorem 7.7. There is a fat domain V containing x_0 and a fat holomorphic function g defined on V such that $g(x_0) \notin \mathcal{A}_{x_0}$.

8 No free implicit function theorem

In this section we prove that the implicit function theorem 6.1 is false in the free category. Indeed, we show that there is a dichotomy: one cannot have an admissible topology τ for which the maps $x \mapsto \|q(x)\|$ are continuous for all $q \in \mathbb{P}^d$ and for which one has both an implicit function theorem (as in the fat and fine topologies) and an affirmative answer to Question 7.1.

Let $p(X, Y, Z)$ be as in (7.3), and define

$$\Phi(X, Y, Z) = (X, Y, p(X, Y, Z)). \tag{8.1}$$

Recall the following condition on solving a Sylvester equation, also called a matrix Ricatti equation [20].

Lemma 8.2. The matrix equation $AH - HB = 0$, for $A, B, H \in \mathbb{M}_n$, has a non-zero solution H if and only if $\sigma(A) \cap \sigma(B) \neq \emptyset$. The dimension of the set of solutions is $\#\{(\lambda, \mu) : \lambda \in \sigma(A), \mu \in \sigma(B), \lambda = \mu\}$, where eigenvalues are counted with multiplicity.

Lemma 8.3. The derivative of Φ, and $\frac{\partial}{\partial Z}p$, are each non-singular if and only if

$$\sigma(X + Y + Z) \cap \sigma(-X - Z) \ = \ \emptyset. \tag{8.4}$$

Proof. $D\Phi$ is non-singular if and only if $\frac{\partial}{\partial Z}p$ is.

$$\begin{aligned} \frac{\partial}{\partial Z}p(X, Y, Z)[H] &= ZH + HZ + XH + HX + YH \\ &= (X + Y + Z)H + (X + Z)H. \end{aligned}$$

The result now follows from Lemma 8.2. □

Lemma 8.5. Let a be as in (7.4). There is a free neighborhood of a on which (8.4) holds; moreover it is of the form G_δ where δ is a diagonal matrix of polynomials.

Proof. The eigenvalues of $a^1 + a^2 + a^3$ are $(1 \pm \sqrt{5})/2$; call them λ_1 and λ_2. The eigenvalues of $a^1 + a^3$ are ± 1. Let $\varepsilon > 0$ be such that the closed disks of radius ε and centers $\lambda_1, \lambda_2, 1, -1$ are disjoint.

Let $\delta(x)$ be the 2-by-2 diagonal matrix with entries

$$M(x^1 + x^2 + x^3 - \lambda_1)(x^1 + x^2 + x^3 - \lambda_2) \text{ and } M(x^1 + x^3 - 1)(x^1 + x^3 + 1).$$

By choosing M large enough, one can ensure that if $x \in G_\delta$, then

$$\sigma(x^1 + x^2 + x^3) \subset \mathbb{D}(\lambda_1, \varepsilon) \cup \mathbb{D}(\lambda_2, \varepsilon) \text{ and } \sigma(x^1 + x^3) \subset \mathbb{D}(1, \varepsilon) \cup \mathbb{D}(-1, \varepsilon).$$

□

Theorem 8.6. Let τ be an admissible topology, defined on $\mathbb{M}^{[d]}$ for all $d \geq 2$. Suppose τ has the property that for each $q \in \mathbb{P}^d$, the map $x \mapsto ||q(x)||$ is τ-continuous from $\mathbb{M}^{[d]}$ to $\mathbb{R}^+$. If every τ holomorphic function is pointwise approximable by free polynomials, then Theorem 6.1 does not hold in the τ category.

If, in addition, τ has the property that the projection maps from $\mathbb{M}^{[d]}$ to $\mathbb{M}^{[d-1]}$ are open, then Theorem 3.2 also does not hold in the τ category.

Proof. Let Φ be as in (8.1) and G_δ as in Lemma 8.5. By Lemma 8.3, $\frac{\partial}{\partial Z} p$ and $D\Phi$ are non-singular on G_δ, and by hypothesis, G_δ is τ-open. If the Implicit function theorem were true for τ, applying it to the set $Z_p \cap G_\delta$, there would be a τ open neighborhood W of $x_0 \in \mathbb{M}_2^2$ and a τ holomorphic function g such that $g(x_0) = z_0$. This cannot occur, because $z_0 \notin \mathcal{A}_{x_0}$.

If the τ Inverse function theorem were true, applying it to the map Φ on G_δ and repeating the proof of Theorem 6.1 would yield the τ Implicit function theorem and the function g. □

Corollary 8.7. Theorem 6.1 does not hold in the free category.

9 Free Algebraic Sets

By a *free algebraic set* in $\mathbb{M}^{[d]}$ we mean the common zero set of some set of free polynomials.

Example 9.1. Consider the polynomial

$$p(X, Y) \;=\; aX^2 + bXY + cYX,$$

where $b \neq -c$, and let $V = Z_p$. The partial derivative with respect to Y is

$$\frac{\partial}{\partial Y} p(X, Y)[H] \;=\; bXH + cHX.$$

By Lemma 8.2, the Sylvester equation $bXH + cHX = 0$ has a non-zero solution if and only if $\sigma(bX) \cap \sigma(-cX)$ is non-empty. Assume that

$$p(X_0, Y_0) = 0, \quad \text{and} \quad \sigma(bX_0) \cap \sigma(-cX_0) = \emptyset. \tag{9.2}$$

Then there is a fat neighborhood of X_0 on which

$$\sigma(bX) \cap \sigma(-cX) \;=\; \emptyset, \tag{9.3}$$

so by Theorem 6.1 there is a function g such that locally $V = \{(X, g(X))\}$. In particular, this forces Y to commute with X, so locally

$$X(aX + (b+c)Y) \;=\; 0.$$

Therefore

$$Y \;=\; -\frac{a}{b+c}X \tag{9.4}$$

since X is invertible by (9.3).

So if (9.2) holds, X_0 and Y_0 commute, and

$$Y_0 \;=\; -\frac{a}{b+c}X_0.$$

Dropping assumption (9.3), how many non-commuting solutions are there? For example, the non-commuting pair

$$\left[\begin{pmatrix} b & 0 \\ 0 & -c \end{pmatrix}, \begin{pmatrix} -\frac{ab}{b+c} & 0 \\ e & \frac{ac}{b+c} \end{pmatrix}\right]$$

satisfies $p(X, Y) = 0$ for any $e \in \mathbb{C}$.

Let k be the number of common eigenvalues of bX and $-cX$, counting multiplicity. For fixed X, the equation

$$bXY - cYX \;=\; -aX^2$$

always has one solution given by (9.4). By Lemma 8.2, it therefore has a k dimensional set of solutions. If X is invertible, the solution from (9.4) is the unique commuting one, so all the others do not commute.

What is the dimension of the set of non-commuting pairs (X, Y) in $\mathbb{M}_n^2$ annihilated by p? If $-b/c$ is a root if unity, it can be larger than n^2. But if $\alpha = -b/c$ is not a root of unity, it is exactly n^2 when $n \geq 2$. Indeed, suppose X has eigenvalues

$$\lambda_1, \alpha\lambda_1, \ldots, \alpha^{k_1}\lambda_1, \lambda_2, \alpha\lambda_2, \ldots, \alpha^{k_2}\lambda_2, \ldots, \lambda_r, \ldots, \alpha^{k_r}\lambda_r$$

with corresponding multiplicities

$$d_{1,0}, d_{1,1}, \ldots, d_{1,k_1}, d_{2,0}, d_{2,1}, \ldots, d_{2,k_2}, \ldots, d_{r,0}, \ldots, d_{r,k_r},$$

where for $i \neq j$, λ_i is not a power of α times λ_j. Then

$$k \;=\; \sum_{i=1}^{r} \sum_{j=1}^{k_i} d_{i,j-1} d_{i,j}.$$

The dimension of the set of X's with this collection of eigenvalues is

$$r + n^2 - \sum_{i=1}^{r} \sum_{j=0}^{k_i} d_{i,j}^2. \tag{9.5}$$

As

$$\sum_{j=1}^{k_i} d_{i,j-1} d_{i,j} + 1 \;\leq\; \sum_{j=0}^{k_i} d_{i,j}^2,$$

we get that the dimension of the set of pairs (X, Y) in Z_p, which is (9.5) plus k, is at most n^2. However, this is attained with $k > 0$ by, for example, choosing $d_{1,0} = d_{1,1} = 1$, and for $i > 1$, choosing $d_{i,0} = 1$ and $d_{i,j} = 0, j \geq 1$.

We summarize:

Proposition 9.6. Assume $b \neq -c$. Let $X_0 \in \mathbb{M}_n$ be fixed and invertible. Let

$$\mathcal{Y} \;=\; \{Y \in \mathbb{M}_n : aX_0^2 + bX_0 Y + cY X_0 = 0\}.$$

Let k be the number of common eigenvalues of bX_0 and $-cX_0$, counting multiplicity.

(i) If $k = 0$, then $\mathcal{Y}$ has a unique element, which commutes with X_0.

(ii) If $k > 0$, then $\mathcal{Y}$ is a k-dimensional affine space in $\mathbb{M}_n$, and it contains a unique element that commutes with X_0.

(iii) If b/c is not a root of unity, then the dimension of the set of non-commuting solutions in $\mathbb{M}_n^2$ of $p(X, Y) = 0$ is exactly n^2 if $n \geq 2$, the same as the dimension of the set of commuting solutions.

The example

$$p(X, Y) = (XY - YX)^2 - \mathrm{id}$$

shows that one can choose a polynomial for which Z_p contains no commuting elements, but for a generic p this does not happen. We can extend this observation to "codimension one" free algebraic sets. For convenience, let us write elements of $\mathbb{M}^{[d]}$ as $(X, Y^1, \ldots, Y^{d-1})$.

Theorem 9.7. Let $k = d - 1$, and let $p_1, \ldots, p_k$ be free polynomials in $\mathbb{P}^d$ with the property that, when evaluated on d-tuples of complex numbers, they are not constant in the last k variables. Let $p = (p_1, \ldots, p_k)^t$, and let

$$V \;=\; \{(X, Y^1, \ldots, Y^k) \;:\; p(X, Y^1, \ldots, Y^k) = 0\}.$$

Let B be the finite (possibly empty) set

$$B \;=\; \cup_{j=1}^{k} \{x \in \mathbb{C} \;:\; \forall y \in \mathbb{C}^k, \;\; p_j(x, y^1, \ldots, y^k) \neq 0\}.$$

If X_0 in $\mathbb{M}_n$ has n linearly independent eigenvectors and $\sigma(X_0) \cap B = \emptyset$, then there exists Y_0 in $\mathbb{M}_n^k$ that satisfies $(X_0, Y_0) \in V$ and such that each element Y_0^j commutes with X_0.

(ii) If (X_0, Y_0) is in V and X_0 and Y_0 do not commute, then we must have

$$(X_0, Y_0) \;\in\; V \cap \{(X, Y) : Dp(X, Y) \text{ is not full rank on } 0 \times \mathbb{M}_n^k\}. \qquad (9.8)$$

Proof. (i) Write X_0 as the diagonal matrix with diagonal entries $(x_1, \ldots, x_n)$ with respect to a basis of eigenvectors. Choose Y_0^j to be the diagonal matrix with diagonal entries $(y_1^j, \ldots, y_n^j)$. Then Y_0^j will commute with X_0; and $p(X_0, Y_0)$ will be zero if $p(x_i, y_i^1, \ldots, y_i^k) = 0$ for each i. This can be done by choosing y_i to be a root of the polynomial $p(x_i, y)$.

(ii) By Theorems 5.5 and 6.1, if Dp is full rank on $0 \times \mathbb{M}_n^k$, then there is a fat holomorphic function g that maps X_0 to Y_0. Since g is a function of one variable, this means Y_0^j is in $\mathcal{A}_{X_0}$ for each j, and so commutes with X_0. $\square$

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