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The Julia-Carathéodory theorem on the bidisk revisited

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Abstract. The Julia quotient measures the ratio of the distance of a function value from the boundary to the distance from the boundary. The Julia-Carathéodory theorem on the bidisk states that if the Julia quotient is bounded along some sequence of nontangential approach to some point in the torus, the function must have directional derivatives in all directions pointing into the bidisk. The directional derivative, however, need not be a linear function of the direction in that case. In this note, we show that if the Julia quotient is uniformly bounded along every sequence of nontangential approach, the function must have a linear directional derivative. Additionally, we analyze a weaker condition, corresponding to being Lipschitz near the boundary, which implies the existence of a linear directional derivative for rational functions.

1. Introduction

Consider the following four holomorphic functions. They all map the bidisk $\mathbb{D}^2$ to $\overline{\mathbb{D}}$, and they all have a singularity at $(1,1)$. However, the nature of the singularity gets progressively worse.

$$\phi_1(z) = \frac{-4z_1z_2^2 + z_2^2 + 3z_1z_2 - z_1 + z_2}{z_2^2 - z_1z_2 - z_1 - 3z_2 + 4},$$

$$\phi_2(z) = \frac{(z_2 - z_1) - 2(1 - z_1)(1 - z_2)\log(\frac{1+z_2}{1-z_2}\frac{1-z_1}{1+z_1})}{(z_2 - z_1) + 2(1 - z_1)(1 - z_2)\log(\frac{1+z_2}{1-z_2}\frac{1-z_1}{1+z_1})},$$

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$$\phi_3(z) = \frac{3z_1z_2 - 2z_1 - z_2}{3 - z_1 - 2z_2},$$
$$\phi_4(z) = \prod_{n=1}^{\text{ß}} \frac{2(1-2^{-n}) - z_1 - z_2}{2 - (1-2^{-n})(z_1+z_2)}.$$

In one variable, the Julia–Carathéodory theorem asserts that for a holomorphic function that maps $\mathbb{D}$ to $\mathbb{D}$, a relatively mild regularity assumption at a boundary point results in seemingly stronger forms of regularity [6,9]. Variants of their results on the bidisk have been studied in [1–5,8,10]. In [3] two notions of regularity, a weaker one called a B point and a stronger one called a C point, were studied (we shall give definitions in Section 2). Letting χ denote the point $(1,1)$ on the distinguished boundary of $\mathbb{D}^2$, the function ϕ_1 above is the only one with a C point at χ; the first three functions have a B point at χ, and ϕ_4 does not.

In this note, we give an alternative characterization of C points (Theorem 2.5). We also introduce a new notion of regularity, called a B+ point, which is intermediate between B point and C point. We analyze these points in Theorem 2.3), and show that the function ϕ_2 above has a B+ point at χ, and ϕ_3 does not. We prove that if a function is rational, every B+ point is actually a C point (Theorem 2.6).

2. B, B+ and C points

In the bidisk, we shall measure distances with the supremum norm. If $\tau \in \partial\mathbb{D}^2$, we shall say that a set $S \subset \mathbb{D}^2$ is non-tangential at τ if there exists a constant $M > 0$ such that

$$\operatorname{dist}(z,\tau) \;\le\; M \operatorname{dist}(z,\partial\mathbb{D}^2) \quad \text{for all} \;\; z \in S. \tag{2.1}$$

The least M that satisfies (2.1) we shall call the *aperture* of S. We shall say that a statement holds non-tangentially at τ if, for every set S that is non-tangential at τ, there exists $\varepsilon > 0$ such that the statement holds on $S \cap \mathbb{B}(\tau,\varepsilon)$.

We shall let Π denote the upper half plane. A Pick function in d variables is a holomorphic function from Π^d to $\overline{\Pi}$.

For the rest of this section, we shall assume that $\varphi\colon \mathbb{D}^2 \to \overline{\mathbb{D}}$ is analytic and $\tau \in \mathbb{T}^2$ is in the distinguished boundary of the bidisk.

The point τ is called a B point for φ if either (A) or (B) of the following equivalent conditions hold:

Theorem 2.1. ([3]) *The following are equivalent:*

(A) $$\liminf_{z\to\tau} \frac{1-|\varphi(z)|}{1-\|z\|} < \infty.$$

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(B) *There exists an* $\omega \in \mathbb{T}$ *such that for every nontangential set* S *there exists* $\alpha > 0, \varepsilon > 0$, *such that for every* $z \in S \cap B(\tau, \varepsilon)$,

$$\|\varphi(z) - \omega\| \leq \alpha \|\tau - z\|. \tag{2.2}$$

Moreover, if these conditions hold, then for every $h \in \mathbb{C}^2$ *such that* $\tau + th \in \mathbb{D}^2$ *for small positive* t, *the directional derivative*

$$D\varphi(\tau)[h] := \lim_{t \downarrow 0} \frac{\phi(\tau + th) - \omega}{t}$$

exists, and is given by

$$D\varphi(\tau)[h] = -\omega\overline{\tau_2} h_2 \ \eta\Big(\frac{\overline{\tau_2} h_2}{\overline{\tau_1} h_1}\Big), \tag{2.3}$$

where $\eta(w)$ *and* $-w\eta(w)$ *are Pick functions in one variable.*

We shall say that a point is a B+ point if the constant α in (2.2) can be chosen independently of the non-tangential set S.

Definition 2.2. We say that φ has a B^+*-point* at τ if there exists an $\omega \in \mathbb{T}$, $\alpha > 0$, such that for every nontangential set S there exists an $\varepsilon > 0$, such that for every $z \in S \cap B(\tau, \varepsilon)$,

$$\|\varphi(z) - \omega\| \leq \alpha \|\tau - z\|.$$

We show the following theorem characterizing B^+ points.

Theorem 2.3. *The following are equivalent:*

(1) φ *has a* B^+ *point at* τ.

(2) φ *has a* B*-point at* τ *and for some constant* α

$$|D\varphi(\tau)[h]| \leq \alpha \|h\|.$$

(3) *There exists an* $\omega \in \mathbb{T}$, *non-negative real numbers* A *and* B, *and a bounded analytic function* g *on the domain* $\mathbb{C} \setminus [-1, 1]$ *such that*

$$D\varphi(\tau)[h] = Ah_1 + Bh_2 + (h_2 - h_1) g\Big(\frac{h_1 + h_2}{h_1 - h_2}\Big).$$

The point τ is called a C point for φ if either condition (C) or (D) below holds. They say that $D\phi(\tau)[h]$ is a linear function of h.

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Theorem 2.4. ([3]) *The following are equivalent:*

(C) *There exist* $\omega \in \mathbb{T}$ *and* $\lambda \in \mathbb{C}^2$ *such that for every nontangential set* S *and every* $\beta > 0$*, there exists* $\varepsilon > 0$ *such that for every* $z \in S \cap B(\tau, \varepsilon)$,

$$\|\varphi(z) - \omega - \lambda \cdot z\| \leq \beta \|\tau - z\|.$$

(D) *There exist* $\omega \in \mathbb{T}$ *and* $\lambda \in \mathbb{C}^2$ *such that* $\lim_{z \xrightarrow{nt} \tau} \varphi(z) = \omega$ *and* $\lim_{z \xrightarrow{nt} \tau} \nabla\phi(z) = \lambda$.

Thus:

- At a B-point, $\varphi(z) = \omega + O(\|\tau - z\|)$ nontangentially.
- At a B^+-point, $\varphi(z) = \omega + O(\|\tau - z\|)$ uniformly nontangentially.
- At a C-point $\varphi(z) = \omega + \eta \cdot z + o(\|\tau - z\|)$ nontangentially.

We prove that if the lim inf in condition (A) is replaced by a non-tangential lim sup, we get another characterization of C points:

Theorem 2.5. *Let* $\varphi \colon \mathbb{D}^2 \to \overline{\mathbb{D}}$ *be analytic and* $\tau \in \mathbb{T}^2$*. If*

$$\limsup_{z \xrightarrow{nt} \tau} \frac{1 - |\varphi(z)|}{1 - \|z\|} < \infty,$$

then φ *has a* C*-point at* τ*. (Here the* lim sup *is over all sequences which approach* τ *nontangentially.)*

We also show that for rational functions, B+ points are C points.

Theorem 2.6. *Let* $\varphi \colon \mathbb{D}^2 \to \overline{\mathbb{D}}$ *be rational. Then if* $\tau \in \partial\mathbb{D}^2$ *is a* $B+$ *point for* φ*, it is a* C *point.*

3. Proofs

In [4, Thm. 5.3], Agler, Tully-Doyle and Young gave the following characterization of the functions η arising in Theorem 2.1. In [4, Thm 6.2], they showed that all such η can arise. (We introduce a factor of 4 in the following theorem just for notational convenience later).

Theorem 3.1. ([4]) *The functions* $\eta(w)$ *and* $-w\eta(w)$ *are both in the Pick class of one variable if and only if there is a positive measure* μ *supported on* $[-1, 1]$ *such that*

$$\eta(w) = -4 \int_{-1}^{1} \frac{1}{(1-t) + (1+t)w} d\mu(t). \tag{3.1}$$

Moreover, for any η *satisfying* (3.1)*, there exists a holomorphic* $\varphi \colon \mathbb{D}^2 \to \mathbb{D}$ *with a* B *point at* χ *satisfying* (2.3)*.*

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Note that any function η given by (3.1) is analytic on $\mathbb{C} \setminus (-\infty, 0]$ and is real-valued on the positive real axis.

A Pick function of two variables is called homogeneous of degree one if $f(wz_1, wz_2) = wf(z_1, z_2)$ whenever $(z_1, z_2), (wz_1, wz_2) \in \Pi^2$. There is a one-to-one correspondence between homogeneous Pick functions of two variables and functions η of the form (3.1).

Proposition 3.2. *Let f be a homogeneous Pick function of degree one in 2 variables. Define η by*

$$\eta(w) = e^{is} f\Big(-\frac{e^{-is}}{w}, -e^{-is} \Big) \tag{3.2}$$

whenever $(-\frac{e^{-is}}{w}, -e^{-is})$ is in Π^2. Then η is well-defined, and both $\eta(w)$ and $-w\eta(w)$ are in the Pick class of one variable. Conversely, if $\eta(w)$ and $-w\eta(w)$ are in the Pick class, then the function f defined by

$$f(z_1, z_2) = \begin{cases} -z_2 \eta(\frac{z_2}{z_1}), & \text{if } \ \frac{z_2}{z_1} \in \Pi \\ -z_2 \overline{\eta(\frac{\bar{z}_2}{\bar{z}_1})}, & \text{if } \ \frac{\bar{z}_2}{\bar{z}_1} \in \Pi \end{cases} \tag{3.3}$$

is a homogeneous Pick function of degree one.

Proof. Let f be given. Homogeneity means (3.2) is independent of s. For any $w \in \Pi$, if s is positive and small enough, the point $(-\frac{e^{-is}}{w}, -e^{-is})$ is in Π^2, and letting $s \downarrow 0$, we get that $\eta(w) \in \Pi$. Similarly, for s positive and sufficiently small, $-w\eta(w) = e^{is/2} f(e^{-is/2}, e^{-is/2} w)$ is in Π.

Conversely, let η be given. Since η is real-valued on the positive real-axis, both definitions of f agree when z_2 is a positive multiple of z_1, so by continuity (3.3) defines f for any $z \in \Pi^2$. By inspection, the formula is homogeneous of degree one. It remains to show that f maps Π^2 to Π.

Let $z \in \Pi^2$, and first assume that $\arg(z_2/z_1) \in [0, \pi)$. Let $\theta_r = \arg(z_r)$, $r = 1, 2$. Then as both $\eta(z_2/z_1)$ and $-(z_2/z_1)\eta(z_2/z_1)$ are in Π, we have

$$0 < \arg \eta(z_2/z_1) < \pi \ \text{ and } \ 0 < \arg \eta(z_2/z_1) + \theta_2 - \theta_1 - \pi < \pi.$$

By (3.3), we have $\arg f(z_1, z_2) = \theta_2 + \arg \eta(z_2/z_1) - \pi$, and this is sandwiched between θ_1 and θ_2. Similarly, if $\arg(z_2/z_1) \in (-\pi, 0]$, we get that $\arg f(z_1, z_2)$ is between θ_2 and θ_1. ∎

Combining Proposition 3.2 with Theorem 3.1, we get that a function f is in the Pick class of two variables and homogeneous of degree one if and only if it can be represented as

$$f(z_1, z_2) = 4 \int_{-1}^{1} \frac{z_1 z_2}{(1+t)z_1 + (1-t)z_2} d\mu(t). \tag{3.4}$$

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Moreover, if φ has a B point at τ, then (2.3) becomes

$$D\varphi(\tau)(i\tau_1 z_1, i\tau_2 z_2) = i\varphi(\tau) f(z_1, z_2). \tag{3.5}$$

We shall rewrite (3.4) in a way that makes it easier to read off regularity.

Lemma 3.3. *If f is of the form (3.4), then f can be written as*

$$f(z_1, z_2) = (\mu_0 - \mu_1)z_1 + (\mu_0 + \mu_1)z_2 + (z_2 - z_1)g\Big(\frac{z_1 + z_2}{z_2 - z_1}\Big), \tag{3.6}$$

where μ_i is the i-th moment of μ and g is a Pick function of one variable such that $\operatorname{Im} g$ *is the Poisson integral of the measure $(1 - t^2)d\mu(t)$.*

Proof. Let us calculate $f(\zeta - 1, \zeta + 1)$ for $\zeta \in \Pi$.

$$\begin{aligned} f(\zeta - 1, \zeta + 1) &= 4\int_{-1}^{1} \frac{(\zeta - 1)(\zeta + 1)}{(1 + t)(\zeta - 1) + (1 - t)(\zeta + 1)} d\mu(t) = 2\int_{-1}^{1} \frac{1 - \zeta^2}{t - \zeta} d\mu(t) \\ &= 2\int_{-1}^{1} \left(\frac{1 - t^2}{t - \zeta} + \frac{t^2 - \zeta^2}{t - \zeta}\right) d\mu(t) = 2\int_{-1}^{1} \left(\frac{1 - t^2}{t - \zeta} + \zeta + t\right) d\mu(t) \\ &= 2\mu_0\zeta + 2\mu_1 + 2\int_{-1}^{1} \frac{1}{t - \zeta}(1 - t^2)d\mu(t). \end{aligned}$$

Define the Pick function g by

$$g(\zeta) = \int_{-1}^{1} \frac{1}{t - \zeta}(1 - t^2)d\mu(t). \tag{3.7}$$

Notice that g is holomorphic on $\mathbb{C} \setminus [-1, 1]$, and satisfies $g(\zeta) = \overline{g(\bar{\zeta})}$. Let $z \in \Pi^2$. If $z_1 \neq z_2$, define $w = \frac{z_2 - z_1}{2}$ and $\zeta = \frac{z_2 + z_1}{z_2 - z_1}$, so $z_1 = w(\zeta - 1)$ and $z_2 = w(\zeta + 1)$. Since z_2 cannot be a negative multiple of z_1, we get that ζ is never in $[-1, 1]$, so (3.6) holds provided $z_1 \neq z_2$. By continuity, it holds for all $z \in \Pi^2$. ∎

Note that from (3.7), we see that $\limsup_{s\to\infty} |isg(is)| < \infty$. Using Lemma 3.3, we obtain that a rational homogeneous Pick function which is linearly bounded must be linear, which in turn will imply that for rational functions, a B^+ point is a C-point.

Lemma 3.4. *Let f be a homogeneous Pick function of degree 1 in two variables represented as in (3.6). There exists a $C > 0$ such that $\|f(z_1, z_2)\| \leq C\|z\|$ if and only if g is a bounded function on $\mathbb{C} \setminus [-1, 1]$. If in addition f is rational, then g is identically 0.*

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Proof. The ($\Leftarrow$) implication is trivial. For the converse, assume f is a homogeneous Pick function of degree 1 in two variables, and represent it as in (3.6). Then we have

$$\begin{aligned} C\|z\| \geq \|f(z_1,z_2)\| &\geq -\|(\mu_0-\mu_1)z_1\| - \|(\mu_0+\mu_1)z_2\| + \left\|(z_2-z_1)g\Big(\frac{z_1+z_2}{z_2-z_1}\Big)\right\| \\ &\geq -|\mu_0-\mu_1|\|z\| - |\mu_0+\mu_1|\|z\| + \left\|(z_2-z_1)g\Big(\frac{z_1+z_2}{z_2-z_1}\Big)\right\|. \end{aligned}$$

So $(C+2\mu_0)\|z\| \geq \left\|(z_2-z_1)g\big(\frac{z_1+z_2}{z_2-z_1}\big)\right\|$. Making the substitution $z=(\zeta-1,\zeta+1)$,we get that

$$(C+2\mu_0)(\|\zeta\|+1) \geq \|2g(\zeta)\| \tag{3.8}$$

for all $\zeta \in \mathbb{C}\setminus[-1,1]$. Since g is the Cauchy transform of a compactly supported measure, it vanishes at ∞, and so (3.8) yields that g is bounded on $\mathbb{C}\setminus[-1,1]$.

If f is rational, then $g(\zeta)=\frac{1}{2}[f(\zeta-1,\zeta+1)]-\mu_0\zeta-\mu_1$, is also rational, and since it vanishes at ∞ and has no poles, it is is identically zero. ∎

Proof of Theorem 2.3. (1) $\Leftrightarrow$ (2): By Theorems 2.1 and 3.1, the derivative of φ at τ is a Pick function of two variables, homogeneous of degree 1, which can therefore be represented as in (3.6). From (3.5), we have that for every h that points into $\mathbb{D}^2$, for $t>0$:

$$\varphi(\tau+th)-\varphi(\tau) = i\varphi(\tau)tf(-i\bar{\tau}h_1,-i\bar{\tau}h_2)+o(t).$$

At a B+ point, the left-hand side is bounded by $\alpha t\|h\|$, which means

$$|D\varphi(\tau)[h]| = |f(-i\bar{\tau}h_1,-i\bar{\tau}h_2)| \tag{3.9}$$

is bounded by $\alpha\|h\|$.

Conversely, assume that there exists an α such that for every direction h pointing into the bidisk at τ, $|D\varphi(\tau)[h]| \leq \alpha\|h\|$, and fix a set S that is non-tangential at τ. Since S is non-tangential, it is enclosed in a cone with apex at τ and some aperture M, so there exists some $\varepsilon>0$ and some compact connected set K of vectors in $\mathbb{C}^2$ with unit norm such that

$$S\cap B(\tau,\varepsilon) \subseteq \{\tau+th:\ 0<t<\varepsilon,\ h\in K\} \subseteq \mathbb{D}^2.$$

For each $0<t<\varepsilon$, let

$$\Psi_t(h)=\frac{\varphi(\tau+th)-\varphi(\tau)}{t}.$$

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The functions $\Psi_t(h)$ are continuous on K, and $\lim_{t\downarrow 0}\Psi_t(h) = D\varphi(\tau)[h]$ exists. Since K is compact, the convergence is uniform. So for all $\beta > 0$, there exists $\delta > 0$ such that

$$|\Psi_t(h) - D\varphi(\tau)[h]| \ < \ \beta \quad \text{for all } \ 0 < t < \delta.$$

Therefore

$$\|\varphi(\tau + th) - \varphi(\tau)| \ \leq \ (\alpha + \beta)t \quad \text{for all } \ 0 < t < \delta,$$

which means

$$\|\varphi(z) - \varphi(\tau)| \ \leq \ (\alpha + \beta)\|\tau - z\| \quad \text{for all } \ z \in S \cap B(\tau, \delta).$$

(2) $\Leftrightarrow$ (3) This follows from (3.9) and Lemma 3.4. ∎

Proof of Theorem 2.6. If φ is rational, then $D\varphi(\tau)[h]$ is a rational function of h, so f is rational. By Theorem 2.3 and Lemma 3.4, if f is rational then g is zero, so f is linear, and hence τ is actually a C point. ∎

We now prove two more lemmata for Theorem 2.5.

Lemma 3.5. *Let f be a homogeneous Pick function of degree 1 in two variables which satisfies*

$$\operatorname{Im} f(z_1, z_2) \leq \gamma \max(\operatorname{Im} z_1, \operatorname{Im} z_2) \tag{3.10}$$

for some $\gamma \geq 0$. Then f is linear.

Proof. Let

$$f(z_1, z_2) = (\mu_0 - \mu_1)z_1 + (\mu_0 + \mu_1)z_2 + (z_2 - z_1)g\Big(\frac{z_1 + z_2}{z_2 - z_1}\Big)$$

as in (3.6), where g is the Cauchy transform of the measure $(1-t^2)d\mu(t)$ on $[-1, 1]$, and $\operatorname{Im} g$ is the Poisson transform of this measure.

We will show g must be equal to 0.

Since $(\mu_0 - \mu_1)z_1 + (\mu_0 + \mu_1)z_2$ satisfies (3.10) for some γ, we get that

$$h(z_1, z_2) = (z_2 - z_1)g\Big(\frac{z_1 + z_2}{z_2 - z_1}\Big)$$

must satisfy, for some perhaps different γ', $\operatorname{Im} h(z_1, z_2) \leq \gamma' \max(\operatorname{Im} z_1, \operatorname{Im} z_2)$. Under the substitution $z_1 = \zeta - 1$, $z_2 = \zeta + 1$ we get that

$$2\operatorname{Im} g(\zeta) \leq \gamma' \operatorname{Im} \zeta. \tag{3.11}$$

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As g is bounded in Π, so is $\operatorname{Im} g$, and as $\operatorname{Im} g$ is the Poisson transform of $(1-t^2)d\mu(t)$, we have that $\operatorname{Im} g(t+iy)dt$ converges weak-star to $(1-t^2)d\mu(t)$ by [7, Theorem I.3.1]. Now, equation (3.11) says that $\operatorname{Im} g(t+iy)dt$ converges weak-star to 0, so $(1-t^2)d\mu(t)$ is the zero measure and hence g is identically 0. ∎

Lemma 3.6. *Suppose*

$$\gamma = \limsup_{z \xrightarrow{nt} \tau} \frac{1-|\varphi(z)|}{1-\|z\|}$$

is finite. The Pick function in two variables $f(z_1,z_2) = -i\overline{\varphi(\tau)}D\varphi(\tau)(i\tau_1 z_1, i\tau_2 z_2)$ *satisfies*

$$\operatorname{Im} f(z_1,z_2) \le \gamma \max(\operatorname{Im} z_1, \operatorname{Im} z_2). \tag{3.12}$$

Proof. Without loss of generality, assume $\tau = \chi = (1,1)$ and $f(\tau) = 1$. Note $\limsup_{z \xrightarrow{nt} \tau} \frac{1-|\varphi(z)|^2}{1-|z_r|^2}$ is finite for $r = 1, 2$. Consider the limit along the ray $z = 1-th$ as $t \downarrow 0$, for any h pointing into $\mathbb{D}^2$. Since we chose $\tau = \chi$, this means $\operatorname{Re}(h_r) < 0$ for $r = 1, 2$, and,

$$\limsup_{z \xrightarrow{nt} \tau} \frac{1-|\varphi(1-th)|^2}{1-|1-th_r|^2} \;\le\; \gamma.$$

So

$$\limsup_{t \to 0^+} \frac{1-(\operatorname{Re}\varphi(1-th))^2-(\operatorname{Im}\varphi(1-th))^2}{1-(\operatorname{Re} 1-th_r)^2-(\operatorname{Im} 1-th_r)^2} \;\le\; \gamma.$$

Applying l'Hospital's rule, we get $\frac{\operatorname{Re} D\varphi(\tau)[h]}{\operatorname{Re} h_r} \le \gamma$. That is, $\operatorname{Re} D\varphi(\tau)[h] \le \gamma \operatorname{Re} h_r$. As $z = ih$, we get (3.12). ∎

Proof of Theorem 2.5. This now follows from Lemmata 3.6 and 3.5. ∎

4. Examples

Consider the functions ϕ_1 to ϕ_4 from the Introduction. Note that $\phi_4(r,r) = B(r)$ where B is the Blaschke product with zeroes at $(1-2^{-n})$. This is an interpolating sequence, so even the limit $\lim_{r\uparrow 1}|\phi_4(r,r)|$ does not exist. In particular, χ fails to be a B point for ϕ_4.

For ϕ_3, we get $\lim_{r\uparrow 1}\frac{1-|\phi(r\chi)|}{1-r} = 1$, so χ is a B point. A calculation yields

$$D\phi_3(\chi)[h] = -\frac{3h_1h_2}{h_1+2h_2}.$$

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Choosing $h = (-\varepsilon + 2i, -\varepsilon - i)$, we see that this is not $O(\|h\|)$, so χ is not a B+ point.

The function ϕ_1 is given in [3] as an example of a function with a C point. It is easy to check that $\phi_1(r,r) = r^2$, so χ is a B point. By a straightforward but lengthier calculation,

$$D\phi_1(\chi)[h] = 2h_2,$$

which is linear, so χ is a C point.

The formula for ϕ_2 reduces to $0/0$ when $z_1 = z_2$; on the diagonal $\phi_2(z)$ should be defined to equal $\frac{-3+5z_1}{5-3z_1}$. We claim that this will give a function in the Schur class that has a B+ point at χ.

To show this, we shall start with g from (3.7), where we choose μ to be linear Lebesgue measure on $[-1,1]$; so

$$g(\zeta) = \int_{-1}^{1} \frac{1-t^2}{t-\zeta} dt = (1-\zeta^2)\log\Big(\frac{\zeta-1}{\zeta+1}\Big) - 2\zeta.$$

The function g is bounded on $\mathbb{C} \setminus [-1,1]$. From (3.6),

$$f(z_1,z_2) = 2z_1 + 2z_2 + (z_2-z_1)g\Big(\frac{z_2+z_1}{z_2-z_1}\Big) = \begin{cases} \frac{-4z_1z_2}{z_2-z_1}\log(\frac{z_1}{z_2}) & z_1 \neq z_2 \\ -4z_1 & z_1 = z_2 \end{cases}.$$

From (3.1), we have

$$\eta(w) = -4\int_{-1}^{1} \frac{dt}{(1-t)+(1+t)w}.$$

Now, we follow the recipe from [4] to produce a function with a B point at χ and slope function η. For each $t \in [-1,1]$, let

$$\psi_t(z) = \Big((1-t)\frac{1+z_1}{1-z_1} + (1+t)\frac{1+z_2}{1-z_2}\Big)^{-1},$$

and let $\psi(z) = 4\int_{-1}^{1} \psi_t(z)dt$. Each ψ_t, and hence also ψ, maps $\mathbb{D}^2$ to the right half-plane. Moreover χ is a B point for ψ, and

$$D\psi(\chi)[h] = -2\int_{-1}^{1} \frac{h_1h_2}{(1-t)h_1 + (1+t)h_2} dt.$$

Finally, we let

$$\phi_2(z) = \frac{1-\psi(z)}{1+\psi(z)} = \begin{cases} \frac{(z_2-z_1)-2(1-z_1)(1-z_2)\log(\frac{1+z_2}{1-z_2}\frac{1-z_1}{1+z_1})}{(z_2-z_1)+2(1-z_1)(1-z_2)\log(\frac{1+z_2}{1-z_2}\frac{1-z_1}{1+z_1})} & z_1 \neq z_2 \\ \frac{-3+5z_1}{5-3z_1} & z_1 = z_2 \end{cases}.$$

Then $\lim_{r\uparrow 1} \phi_2(r,r) = 1$ and $D\phi_2(\chi)[h] = -2D\phi(\chi)[h] = -h_2\eta(h_2/h_1)$, so ϕ_2 has the required slope function, and hence χ is a B+ point.

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