TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Xxxx XXXX, Pages 000–000
S 0002-9947(XX)0000-0

NORM PRESERVING EXTENSIONS OF BOUNDED HOLOMORPHIC FUNCTIONS

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Abstract. A relatively polynomially convex subset V of a domain Ω has the extension property if for every polynomial p there is a bounded holomorphic function ϕ on Ω that agrees with p on V and whose H^∞ norm on Ω equals the sup-norm of p on V. We show that if Ω is either strictly convex or strongly linearly convex in $\mathbb{C}^2$, or the ball in any dimension, then the only sets that have the extension property are retracts. If Ω is strongly linearly convex in any dimension and V has the extension property, we show that V is a totally geodesic submanifold. We show how the extension property is related to spectral sets.

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1. Introduction

1.1. **Statement of results.** Let Ω be an open set in $\mathbb{C}^d$, and let V be a subset of Ω, not necessarily open. A function $f : V \to \mathbb{C}$ is said to be *holomorphic* if, for every point $\lambda \in V$, there exists $\varepsilon > 0$ and a holomorphic function F defined on the ball $B(\lambda, \varepsilon)$ in $\mathbb{C}^d$ such that F agrees with f on $V \cap B(\lambda, \varepsilon)$. Let $H^\infty(V)$ denote the algebra of all bounded holomorphic functions on V, equipped with the sup-norm on V. Let $\mathcal{A}$ be a subalgebra of $H^\infty(V)$, with the same norm.

Definition 1.1. We say that V has the $\mathcal{A}$ *extension property* if, for every f in $\mathcal{A}$, there exists $F \in H^\infty(\Omega)$ such that $F|_V = f$ and $\|F\|_\Omega = \|f\|$. If Ω is a bounded domain and $\mathcal{A}$ is the algebra of polynomials, we shall say that V has the extension property.

If Ω is pseudo-convex and V is an analytic subvariety of Ω, it is a deep theorem of Cartan that every holomorphic function on V extends to a holomorphic function on Ω [10]. However, in general, functions do not have extensions that preserve the H^∞-norm. There is one easy way to have a norm preserving extension. We say V is a *retract* of Ω if there is a holomorphic map $r : \Omega \to \Omega$ such that the range of r is V and $r|_V$ is the identity. If V is a retract, then $f \circ r$ will always be a norm preserving extension of f to Ω.

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In [4], it was shown that if Ω is the bidisk, that is basically the only way that sets can have the extension property.

Theorem 1.2 ([4])**.** *Let V be a relatively polynomially convex subset of $\mathbb{D}^2$. Then V has the extension property if and only if V is a retract of $\mathbb{D}^2$.*

Received by the editors MONTH DAY, YEAR.
2010 *Mathematics Subject Classification.*
The first author was partially supported by the NCN Grant UMO-2014/15/D/ST1/01972.
The second author was partially supported by National Science Foundation Grant DMS 156243.

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We say that a set V contained in a domain Ω is *relatively polynomially convex* if the intersection of the polynomial hull $\hat{V}$ with Ω is V. If $\hat{V} \cap \Omega$ has the extension property, so does V, so the assumption of relative polynomial convexity is a natural one.

In [1], Agler, Lykova, and Young proved that, for the symmetrized bidisk, that is, the set

$$G \;=\; \{(z+w, zw) : z, w \in \mathbb{D}\}, \tag{1.3}$$

not all sets with the extension property are retracts. They proved the following.

Theorem 1.4 ([1])**.** *The set V is an algebraic subset of G having the $H^\infty(V)$ extension property if and only if either V is a retract of G or $V = \mathcal{R} \cup \mathcal{D}_\beta$, where $\mathcal{R} = \{(2z, z^2) : z \in \mathbb{D}\}$ and $\mathcal{D}_\beta = \{(\beta + \bar{\beta}z, z) : z \in \mathbb{D}\}$, and $\beta \in \mathbb{D}$.*

It is the purpose of this note to study the extension property for domains other than the bidisk and symmetrized bidisk. Our first main result is for strictly convex bounded domains in $\mathbb{C}^2$.

Theorem 1.5. *Let Ω be a strictly convex bounded subdomain of $\mathbb{C}^2$, and assume that $V \subseteq \Omega$ is relatively polynomially convex. Then V has the extension property if and only if V is a retract of Ω.*

We prove Theorem 1.5 in Section 2. In Section 3, we prove a similar result for balls in any dimension. In Section 4, we consider strongly linearly convex domains. We shall give a definition of strongly linearly convex in Section 4; roughly, it says that the domain does not have second order contact with complex tangent planes. We prove the following.

Theorem 1.6. *Let $\Omega \subseteq \mathbb{C}^d$ be a strongly linearly convex bounded domain with $\mathcal{C}^3$ boundary, and assume that $V \subseteq \Omega$ is relatively polynomially convex, and not a singleton. If V has the extension property, then V is a totally geodesic complex submanifold of Ω.*

As a corollary of Theorem 1.6, we conclude the following.

Corollary 1.7. *Let $\Omega \subseteq \mathbb{C}^2$ be a strongly linearly convex bounded domain with $\mathcal{C}^3$ boundary, and assume that $V \subseteq \Omega$ is relatively polynomially convex. If V has the extension property, then V is a retract.*

In [23], Pflug and Zwonek proved that the symmetrized bidisk is an increasing union of strongly linearly convex domains with smooth (even real analytic) boundaries. So contrasting Theorem 1.4 with Corollary 1.7 shows that the extension property implying a retract is not stable under increasing limits.

1.2. **Motivation and history.** One reason to study sets with the extension property is provided by spectral sets. Let Ω be an open set in $\mathbb{C}^d$. If $T = (T_1, \ldots, T_d)$ is a d-tuple of commuting operators on a Hilbert space with spectrum in Ω, then one can define $f(T)$ for any $f \in H^\infty(\Omega)$ by the Taylor functional calculus [25]. We say Ω is a *spectral set* for T if the following analogue of von Neumann's inequality holds:

$$\|f(T)\| \;\leq\; \sup_{\lambda \in \Omega} |f(\lambda)| \qquad \forall f \in H^\infty(\Omega). \tag{1.8}$$

If $V \subseteq \Omega$, we say that T is *subordinate to* V if the spectrum of T is in V and $f(T) = 0$ whenever $f \in H^\infty(\Omega)$ and $f|_V = 0$.

Definition 1.9. Let $V \subseteq \Omega$, a domain in $\mathbb{C}^d$. Let $\mathcal{A}$ be a subalgebra of $H^\infty(V)$ that contains the polynomials and has the property that every f in $\mathcal{A}$ can be extended to some function ϕ in $H^\infty(\Omega)$ that agrees with f on V. We say that V is an $\mathcal{A}$ von Neumann set with respect to Ω if, whenever T is a d-tuple of commuting operators on a Hilbert space that has Ω as a spectral set and that is subordinate to V, then Q4

$$\|f(T)\| \leq \sup_{\lambda \in V} |f(\lambda)| \qquad \forall f \in \mathcal{A}. \tag{1.10}$$

To make sense of (1.10), we must know what we mean by $f(T)$. Since every f in $\mathcal{A}$ can be extended to some ϕ in $H^\infty(\Omega)$, we can define $f(T) = \phi(T)$. Since T is subordinate to V, the definition does not depend on the choice of extension—if ϕ_1 and ϕ_2 are both extensions of f that are in $H^\infty(\Omega)$, then $\phi_1(T) = \phi_2(T)$ since $\phi_1 - \phi_2$ is identically 0 on V.

Note the big difference between (1.8) and (1.10) is whether the norm of $f(T)$ is controlled by just the values of f on V or all of the values on Ω.

One of the main results of [4] is as follows.

Theorem 1.11. *Let Ω be the bidisk $\mathbb{D}^2$, and let V be a subset. Let $\mathcal{A}$ be a subalgebra of $H^\infty(V)$ that contains the polynomials. Then V is an $\mathcal{A}$ von Neumann set if and only if it has the $\mathcal{A}$ extension property.*

In [1], the same theorem is proved when Ω is the symmetrized bidisk (1.3). In Section 5, we prove that the theorem holds for any bounded domain Ω and any algebra containing the polynomials.

Theorem 1.12. *Let Ω be a bounded domain in $\mathbb{C}^d$, and let $V \subseteq \Omega$. Let $\mathcal{A}$ be a subalgebra of $H^\infty(V)$ that contains the polynomials. Then V is an $\mathcal{A}$ von Neumann set if and only if it has the $\mathcal{A}$ extension property.*

Another reason to study sets with the extension property is if one wishes to understand Nevanlinna–Pick interpolation. Given a domain Ω and N distinct points $\lambda_1, \ldots, \lambda_N$ in Ω, the Nevanlinna–Pick problem is to determine, for each given set $w_1, \ldots, w_N$ of complex numbers,

$$\inf\{\|\phi\|_{H^\infty(\Omega)} \ : \ \phi(\lambda_i) = w_i, \ 1 \leq i \leq N\},$$

and to describe the minimal norm solutions. This problem has been extensively studied in the disk [9], where the minimal norm solution is always unique, but is more elusive in higher dimensions. Tautologically there is some holomorphic subvariety V on which all minimal norm solutions coincide, but sometimes one can actually say something descriptive about V, as in [3, 17]. If V had the extension property, one could split the analysis into two pieces: finding the unique solution on V, and then studying how it extends to Ω.

The first result we know of norm preserving extensions is due to Rudin [24, Thm. 7.5.5], who showed that if V is an embedded polydisk in Ω and there is an extension operator from $H^\infty(V)$ to $H^\infty(\Omega)$ that is linear of norm 1, then V must be a retract of Ω. Theorem 1.2 from [4] characterized sets in $\mathbb{D}^2$ that have the extension property, and Theorem 1.4 from [1] did this for the symmetrized bidisk. Neither of these results assume that there is a linear extension operator. In [15], Guo, Huang, and Wang proved the following.

Theorem 1.13. *Suppose that V is an algebraic subvariety of $\mathbb{D}^3$ that has the $H^\infty(V)$ extension property, and that there is a linear extension operator from $H^\infty(V)$ to $H^\infty(\mathbb{D}^3)$. Then V is a retract of $\mathbb{D}^3$.*

If V is an $H^\infty(\mathbb{D}^d)$-convex subset[1] *of $\mathbb{D}^d$ for any d, it has the $H^\infty(V)$ extension property, and there is a norm 1 extension operator that is an algebra homomorphism, then V is a retract of $\mathbb{D}^d$.*

2. Strictly convex domains in $\mathbb{C}^2$

A convex set Ω in $\mathbb{C}^d$ is called *strictly convex* if for every boundary point λ, there is a real hyperplane P such that $P \cap \overline{\Omega} = \{\lambda\}$. Equivalently, it means that there are no line segments in $\partial\Omega$.

Let Ω be a domain in $\mathbb{C}^d$, and let λ, μ be two distinct points in Ω. Following [1], we shall call the pair $\delta = (\lambda, \mu)$ a *datum*. A *Kobayashi extremal* for δ is a holomorphic map $k : \mathbb{D} \to \Omega$ such that both λ and μ are in the range of k, and the pseudo-hyperbolic distance ρ between the preimages of λ and μ is minimized at k over all holomorphic maps from $\mathbb{D}$ to Ω. A *Carathéodory extremal* for δ is a holomorphic map $\phi : \Omega \to \mathbb{D}$ that maximizes $\rho(\phi(\lambda), \phi(\mu))$.

If δ is a datum, we shall say that the Kobayashi extremal is *essentially unique* if, given any two Kobayashi extremals k_1 and k_2, they are related by $k_2 = k_1 \circ m$, where m is a Möbius automorphism of $\mathbb{D}$. We shall call the range of a Kobayashi extremal with datum δ a *geodesic* through δ. If a set $V \subseteq \Omega$ has the property that for any λ, μ in V, a geodesic through (λ, μ) is contained in V, we shall say that V is *totally geodesic*.

A theorem of Lempert [20] asserts that if Ω is a bounded convex domain, every Kobayashi extremal k has a left inverse, i.e., a Carathéodory extremal ϕ satisfying

$$\phi(k(z)) \;=\; z \qquad \forall z \in \mathbb{D}.$$

A consequence is that every geodesic is a retract since $r = k \circ \phi$ is the identity on $\mathrm{Ran}(r)$. By Lempert's theorem, on a convex domain the Kobayashi distance between two points λ and μ in Ω is the same as the Carathéodory distance, defined by

$$\kappa(\lambda, \mu) \;=\; \rho(\phi(\lambda), \phi(\mu)),$$

where ϕ is a Carathéodory extremal for the datum (λ, μ).

Much of the difficulty in characterizing subsets of $\mathbb{D}^2$ with the extension property in [4] stemmed from the fact that on the bidisk, Kobayashi extremals need not be essentially unique. In strictly convex domains, Kobayashi extremals are essentially unique [16, Prop. 8.3.3]. Kobayashi extremals are also essentially unique in the symmetrized bidisk G [5].

We shall need the following result, the Royden–Wong theorem. A complete proof is in [16, Lemmata 8.2.2 and 8.2.4 and Remark 8.2.3]. For a function on the unit disk that is in a Hardy space, we shall use the same symbol for the function on $\mathbb{D}$ and for its nontangential limit function on the circle $\mathbb{T}$. We shall use "$\bullet$" to mean the bilinear form on $\mathbb{C}^d$,

$$z \bullet w \;=\; \sum_{j=1}^{d} z_j w_j.$$

Theorem 2.1. *Let Ω be a bounded convex domain in $\mathbb{C}^d$, and let $k : \mathbb{D} \to \Omega$ be a Kobayashi extremal for some datum. Then*

(i) *$k(z) \in \partial\Omega$ for a.e. $z \in \mathbb{T}$.*

[1] We say V is $H^\infty(\mathbb{D}^d)$-convex if for every point λ in $\mathbb{D}^d \setminus V$, there is a function ϕ in $H^\infty(\mathbb{D}^d)$ such that $|\phi(\lambda)| > \sup_{z \in V} |\phi(z)|$.

(ii) *There exists a nonzero function* $h \in H^1(\mathbb{D}, \mathbb{C}^d)$ *such that*

$$\mathrm{Re}\,[(\lambda - k(z)) \bullet (\bar{z}h(z))] < 0 \qquad \forall \lambda \in \Omega, \text{ and a.e. } z \in \mathbb{T}. \tag{2.2}$$

(iii) *There exists a holomorphic* $\phi : \Omega \to \mathbb{D}$ *that satisfies that* $\phi \circ k$ *is the identity on* $\mathbb{D}$ *and satisfies the equation*

$$[\lambda - k(\phi(\lambda))] \bullet h(\phi(\lambda)) \;=\; 0 \qquad \forall \lambda \in \Omega.$$

Definition 2.3. Let Ω be an open set in $\mathbb{C}^d$, and let $V \subseteq \Omega$. Let $\overline{V}$ denote the closure of V in $\mathbb{C}^d$. We say that V is relatively polynomially convex if $\overline{V} \cap \Omega = V$ and $\overline{V}$ is polynomially convex in $\mathbb{C}^d$.

Proposition 2.4. *Let* Ω *be a strictly convex bounded domain in* $\mathbb{C}^d$*. Let* V *be relatively polynomially convex in* Ω*. If* V *has the extension property, then* V *is totally geodesic.*

Proof. Let us assume that μ, μ' are distinct points in V, let $\mathcal{G}$ be the unique geodesic through them, and let $k : \mathbb{D} \to \mathcal{G} \subset \Omega$ be a Kobayashi extremal for these points. Assume that $\mathcal{G}$ is not contained in $\overline{V}$; we shall derive a contradiction.

Since $\overline{V}$ is polynomially convex, there must exist some part of the boundary of $\mathcal{G}$ that is not in $\overline{V}$. So there exists $\xi \in \partial\mathcal{G},\ \eta \in \mathbb{T}$, and $\varepsilon_0, \varepsilon_1 > 0$, so $B(\xi, \varepsilon_0) \cap \overline{V} = \emptyset$ and $k(\Delta(\eta, \varepsilon_1)) \subseteq B(\xi, \varepsilon_0)$, where $\Delta(\eta, \varepsilon_1)$ is some triangle in $\mathbb{D}$ with vertex η and diameter ε_1.

Let h and ϕ be as in Theorem 2.1. Wiggling η a little if necessary, we can assume that both h and k have nontangential limits at η, and that (2.2) holds for $z = \eta$. Then by part (ii) of Theorem 2.1, we have that

$$\{\lambda \in \Omega : \mathrm{Re}\,[(\lambda - \xi) \bullet (\bar{\eta}h(\eta))] = 0\}$$

is a supporting plane for Ω that contains ξ. Since Ω is strictly convex, small perturbations of this plane will intersect Ω only in $B(\xi, \varepsilon_0)$. So there is a small triangle $\Delta(\eta, \varepsilon_2)$ such that for $z \in \Delta(\eta, \varepsilon_2)$, we have

$$\{\lambda \in \Omega : \mathrm{Re}\,[(\lambda - k(z)) \bullet \overline{z}h(z)] = 0\} \cap V \;=\; \emptyset. \tag{2.5}$$

Therefore if $\lambda \in V$ and $z = \phi(\lambda)$, then by part (iii) of Theorem 2.1,

$$[\lambda - k(z)] \bullet h(z) \;=\; 0. \tag{2.6}$$

If z were in $\Delta(\eta, \varepsilon_2)$, then by (2.5), we would have

$$\mathrm{Re}\,[(\lambda - k(z)) \bullet \overline{z}h(z)] \neq 0,$$

which would contradict (2.6). So we can conclude that if $\lambda \in V$, then $\phi(\lambda) \notin \Delta(\eta, \varepsilon_2)$.

The set Ω is convex, and without loss of generality we can assume $0 \in \Omega$. Let $\phi_t(\lambda) = \phi(t\lambda)$ for t in the interval $[0, 1]$. By continuity, there exists $t_0 < 1$ and a nonempty triangle $\Delta(\eta, \varepsilon_3)$, so, for every t between t_0 and 1,

$$\phi_t(V) \cap \Delta(\eta, \varepsilon_3) \;=\; \emptyset. \tag{2.7}$$

Since $\phi(\mu) \neq \phi(\mu')$ (as $\phi \circ k$ is the identity), increasing t_0 we can also assume that $\phi_t(\mu) \neq \phi_t(\mu')$ for $t \in [t_0, 1]$. Now we adapt an idea of Thomas [26]. Let g be the Riemann map from $\mathbb{D} \setminus \Delta(\eta, \varepsilon_3)$ to $\mathbb{D}$. Let $\psi_t = g \circ \phi_t$. For any t in $(t_0, 1)$, the function ϕ_t is holomorphic on a neighborhood of $\overline{\Omega}$, and by (2.7) there is a

neighborhood of $\overline{V}$ on which ψ_t is defined. Since g maps a subset of the disk to the whole disk, there is some $\varepsilon_4 > 0$, independent of t, such that

$$\rho(g(\phi_t(\mu)), g(\phi_t(\mu'))) \ > \ \rho(\phi_t(\mu), \phi_t(\mu')) + \varepsilon_4.$$

By the Oka–Weil theorem, we can approximate ψ_t uniformly by polynomials on V, and hence by the extension property, we can find a holomorphic function F_t that maps Ω to $\mathbb{D}$ and such that

$$\rho(F_t(\mu), \psi_t(\mu)) + \rho(F_t(\mu'), \psi_t(\mu')) \ < \ \frac{\varepsilon_4}{2}.$$

Therefore

$$\rho(F_t(\mu), F_t(\mu')) \ > \ \rho(\phi_t(\mu), \phi_t(\mu')) + \frac{\varepsilon_4}{2}.$$

Since $\phi_t(\lambda)$ approaches $\phi(\lambda)$ for $\lambda \in \Omega$, as $t \to 1$, taking t sufficiently close to 1, we have that $F := F_t$ is a holomorphic function $\Omega \to \mathbb{D}$ that satisfies

$$\rho(F(\mu), F(\mu')) \ > \ \rho(\phi(\mu), \phi(\mu')). \tag{2.8}$$

Inequality (2.8) means that ϕ is not a Carathéodory extremal for (μ, μ') in Ω. But this is a contradiction since ϕ is a left inverse to k. □

We can now prove Theorem 1.5.

Proof of Theorem 1.5. Assume that V has the extension property. It is obviously a retract if V is a singleton, so we shall assume that it contains at least two points. By Proposition 2.4, we know that V is totally geodesic. We shall prove that in fact V must either be a single geodesic, and hence a one-dimensional retract, or all of Ω.

Take two different points in V, and let $\mathcal{G}$ be a unique geodesic of Ω passing through them. Since V is totally geoedesic, $\mathcal{G}$ is contained in V. Assume that there is some point $a \in V \setminus \mathcal{G}$. For each point $\lambda \in \mathcal{G}$, let k_λ be the Kobayashi extremal that passes through a and λ, normalized by $k_\lambda(0) = a$ and $k_\lambda(r) = \lambda$ for some r in the interval $(0, 1)$; so by Lempert's theorem [20], r is $\kappa(\lambda, a)$, the Kobayashi distance between λ and a.

Let $\mathcal{D}$ be a subdisk of $\mathcal{G}$ with compact closure. Note that

$$\{r : k_\lambda(r) = \lambda, \ \lambda \in \overline{\mathcal{D}}\} = \{\kappa(\lambda, a) : \ \lambda \in \overline{\mathcal{D}}\}$$

is bounded away from 0 and 1, as Ω is a bounded domain. Let $\mathcal{B} = \mathbb{D}(\frac{1}{2}, \frac{1}{4})$. Define

$$\begin{aligned} \Psi : \mathcal{B} \times \mathcal{D} &\to \Omega, \\ (z, \lambda) &\mapsto k_\lambda(z). \end{aligned}$$

□

Claim. Ψ is continuous and injective.

Proof of claim. Since geodesics are unique in strictly convex domains [16, Prop. 8.3.3], any two geodesics that share two points must coincide. As $k_\lambda(0) = a$ for each λ, it follows that if $\lambda_1 \neq \lambda_2$, then $k_{\lambda_1}(z_1) \neq k_{\lambda_2}(z_2)$, unless $z_1 = 0 = z_2$. Moreover, since each k_λ has a left inverse, each k_λ is injective, so Ψ is injective.

To see that Ψ is continuous, suppose that $\lambda_n \to \lambda$ and $z_n \to z$. We have $r_n = \kappa(\lambda_n, a)$ satisfies $k_{\lambda_n}(r_n) = \lambda_n$. By Montel's theorem, every subsequence of k_{λ_n} has a subsequence that converges uniformly on compact subsets of $\mathbb{D}$ to a function $k : \mathbb{D} \to \Omega$ (since Ω is bounded and convex, and hence taut). But clearly $k(0) = a$ and k maps some positive real number r to λ, where $r = \lim \kappa(\lambda_n, a) = \kappa(\lambda, a)$ (use a uniform convergence argument together with the equality $k_{\lambda_n}(r_n) = \lambda$). So k is a holomorphic map from $\mathbb{D}$ to Ω that maps 0 to a, and r to λ, and is therefore

a Kobayashi extremal for the datum (λ, a). Since these are essentially unique in strictly convex domains, we have that $k = k_\lambda$. So $\lim \Psi(\lambda_n, z_n) = \Psi(\lambda, z)$, and hence Ψ is continuous. $\lhd$

So Ψ is a continuous injective map between two open subsets of $\mathbb{C}^2$; hence by the invariance of domain theorem, Ψ is open. Therefore the range of Ψ is an open subset U of Ω. As U is a union of geodesics going through pairs of points in V (more precisely, through $a \in V$ and $\lambda \in \mathcal{D} \subset V$) and V is totally geodesic by Proposition 2.4, we have that V contains the whole open set U. This forces V to be all of Ω since, if ν is any point in U and μ is any point not in U, there is some geodesic containing ν and μ. As U is open, this geodesic must contain a continuum of points in U. In particular, it contains two distinct points in $U \subseteq V$, so as V is totally geodesic, we have that V contains the whole geodesic, and in particular $\mu \in V$. Therefore $V = \Omega$. $\square$

3. The ball

Let $\mathbb{B}_d$ be the unit ball in $\mathbb{C}^d$, the set $\{z \in \mathbb{C}^d : \sum_{j=1}^d |z_j|^2 < 1\}$.

Theorem 3.1. *Let V be a relatively polynomially convex subset of $\mathbb{B}_d$ that has the extension property. Then V is a retract of $\mathbb{B}_d$.*

Proof. The result is obvious if V is a singleton, so let us assume it has more than one point. Composing with an automorphism of $\mathbb{B}_d$, we can assume that $0 \in V$. We will show that then V is the image under a unitary map of $\mathbb{B}_k$ for some $1 \leq k \leq d$. First observe that if $a \in V \setminus \{0\}$, we can compose with a unitary so that it has the form $(a_1, 0, \ldots, 0)$. By Proposition 2.4, we have that $\mathbb{B}_1 \subseteq V$. Now we proceed by induction. Suppose that $\mathbb{B}_{k-1} \subsetneq V$, and that $b \in V \setminus \mathbb{B}_{k-1}$. Composing with a unitary, we can assume that $b = (b_1, \ldots, b_k, 0, \ldots, 0)$ and $b_k \neq 0$. Let c be any point in $\mathbb{B}_k$ with $\|c\| < |b_k|/2$. Then

$$c \;=\; \frac{c_k}{b_k}(b_1, \ldots, b_k, 0, \ldots, 0) \;+\; (c_1 - \frac{c_k}{b_k}b_1, c_2 - \frac{c_k}{b_k}b_2, \ldots, 0, \ldots, 0).$$

Then the first point on the right-hand side is in the geodesic connecting b and 0, so it is in V; and the second point is in $\mathbb{B}_{k-1}$, so it is also in V. Therefore the geodesic containing these two points, which is the intersection of the plane containing these two points with $\mathbb{B}_d$, is also in V, and hence $c \in V$.

Continuing until we exhaust V, we conclude that $V = \Phi(\mathbb{B}_k)$ for some k between 1 and d and some automorphism Φ of $\mathbb{B}_d$. $\square$

4. Strongly linearly convex domains

A domain $\Omega \subseteq \mathbb{C}^d$ is called *linearly convex* if, for every point $a \in \mathbb{C}^d \setminus \Omega$, there is a complex hyperplane that contains a and is disjoint from Ω. Now assume that Ω is given by a $\mathcal{C}^2$ defining function r (i.e., $\Omega = \{z : r(z) < 0\}$ and $\operatorname{grad}(r) \neq 0$ on $\partial\Omega$). Then Ω is called *strongly linearly convex* if

$$\sum_{j,k=1}^d \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(a) X_j \bar{X}_k \quad > \quad |\sum_{j,k=1}^d \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(a) X_j X_k|$$

$$\forall a \in \partial\Omega,\ X \in (\mathbb{C}^d)_* \text{ with } \sum_j^d \frac{\partial r}{\partial z_j}(a) X_j = 0.$$

Notice that we check the inequality only on complex tangent vectors. Roughly speaking, a domain is strongly linearly convex if it is smooth and linearly convex and it remains linearly convex after small deformations.

A smooth domain D is strictly convex if for any a in the boundary of D its defining function restricted to the real tangent plane to ∂D at a attains a strict minimum at a, and it is strictly linearly convex if for any a in the boundary of D its defining function restricted to the complex tangent plane (i.e., the biggest complex plane contained in the tangent plane) to ∂D at a attains a strict minimum at a. Note that a smooth domain is strongly linearly convex if the Hessian of its defining function is strictly positive on the complex tangent plane. We call a smooth domain strongly convex if the Hessian of its defining function is strictly positive on the real tangent plane. It is obvious that strong linear convexity implies strict linear convexity, and that any strongly convex domain is strongly linearly convex.

For convenience we shall consider only $\mathcal{C}^3$ domains, though the regularity actually needed is $\mathcal{C}^{2,\epsilon}$ (which means that second order derivatives of the defining function are ϵ-Hölder continuous).

If Ω is strongly linearly convex and has a smooth boundary, it was proved by Lempert [21] that the Kobayashi extremals are unique and depend smoothly on points, vectors, and even domains (in the sense of their defining functions).

Lemma 4.1 ([21, Proposition 11]; see also [19, Proof of Theorem 3.1])**.** *Let Ω be a strongly linearly convex domain with $\mathcal{C}^3$ boundary, and let $f : \mathbb{D} \to \Omega$ be a complex geodesic. Then there exist a domain $G \subset \mathbb{D} \times \mathbb{C}^{d-1}$ and a biholomorphic mapping $\Gamma : \Omega \to G$ that extends to a homeomorphism $\overline{D} \to \overline{G}$ and such that $\Gamma(f(\lambda)) = (\lambda, 0)$, $\lambda \in \mathbb{D}$. Moreover, $\overline{G} \cap (\mathbb{T} \times \mathbb{C}^{d-1}) = \mathbb{T} \times \{0\}$.*

Proof of Theorem 1.6. The proof is split into two parts. The first one says that V is totally geodesic. To get it, we shall use the argument from the proof of Proposition 2.4, but by using Lemma 4.1 in lieu of the Royden–Wong theorem. □

The second part says that any totally geodesic variety in Ω having an extension property is regular.

Lemma 4.2. *Let Ω and V be as in Theorem* 1.6*. Then V is totally geodesic.*

Proof. It was proven by Lempert in [21] for smoothly bounded strongly linearly convex domains, and in [18] for $\mathcal{C}^2$-boundaries, that the Carathéodory and Kobayashi metrics coincide on Ω, and that geodesics $f : \mathbb{D} \to \Omega$ are essentially unique and $\mathcal{C}^{1/2}$-smooth on $\mathbb{D}$, so they extend continuously to the closed unit disk.

Take $\mu, \mu' \in V$ and a complex geodesic f passing through these points. Let $\mathcal{F} = f(\mathbb{D})$, and let Γ be as in Lemma 4.1, a biholomorphic mapping that maps Ω to $G \subset \mathbb{D} \times \mathbb{C}^{d-1}$ that extends continuously to $\overline{\Omega}$ and satisfies $\Gamma(f(\lambda)) = (\lambda, 0)$. Define $\pi(x) = x_1$ for $x = (x_1, \dots, x_d) \in \mathbb{C}^d$. Then $F := \pi \circ \Gamma$ is a left inverse of f and extends continuously on $\overline{\Omega}$.

We shall show that the set $F(\overline{V})$ contains the whole circle $\mathbb{T}$. Assume that this is not true. Then $F(\overline{V})$ omits some $\overline{\mathbb{D}(\xi, \varepsilon_0)} \cap \overline{\mathbb{D}}$ for $\xi \in \mathbb{T}$ and $\varepsilon_0 > 0$. As Ω is strongly pseudo-convex (as a strongly linearly convex domain), the function F (holomorphic on Ω and continuous on $\overline{\Omega}$) can be uniformly approximated by functions that are holomorphic on neighborhoods of $\overline{\Omega}$ [11, Thm. ii.5.10]. Since strong linear convexity is preserved under small perturbations, we can assume that the domains where the approximating functions are defined are also strongly linearly convex. Any

smooth strongly linearly convex domain is Runge (see, for example, [8, 2.1.9, 2.3.9, and 2.5.18]), so any function that is holomorphic on a neighborhood of $\overline{\Omega}$ can be uniformly approximated on Ω by polynomials. Therefore we can find a sequence of polynomials (p_n) such that $p_n(\overline{V}) \subset \mathbb{D} \setminus D(\xi, \varepsilon_0)$ and p_n converges uniformly to F on $\overline{V}$. Since V has the extension property, we can extend each polynomial p_n to an $H^\infty(\Omega)$ functions $\phi_n : \Omega \to \mathbb{D} \setminus \mathbb{D}(\xi, \varepsilon_0)$. Let $g : \mathbb{D} \setminus \mathbb{D}(\xi, \varepsilon_0) \to \mathbb{D}$ be the Riemann map.

There exist $\varepsilon_1 > 0$ and $N \in \mathbb{N}$ such that

$$\rho(g(\phi_n(\mu)), g(\phi_n(\mu'))) \ > \rho(\phi_n(\mu), \phi_n(\mu')) + \varepsilon_1$$

for all $n \geq N$. Thus for n big enough, we get that

$$\rho(g(\phi_n(\mu)), g(\phi_n(\mu'))) \ > \rho(F(\mu), F(\mu')).$$

This contradicts the fact that F is a left inverse to the geodesic f, and thus a Carathéodory extremal for (μ, μ').

We have shown that $\mathbb{T} \subset F(\overline{V})$. By Lemma 4.1, the only points of $\overline{\Omega}$ on which $F \circ \Gamma$ is unimodular are on the boundary of $\mathcal{F}$. Therefore, since V is relatively polynomially convex, we have $\mathcal{F} \subseteq V$, as required. □

Lemma 4.3. *Let Ω be a bounded open set in $\mathbb{C}^d$, and let $V \subseteq \Omega$ be a relatively polynomially convex set that has the extension property. Then V is a holomorphic subvariety of Ω.*

Proof. Let b be any point in $\Omega \setminus V$. Then there is a polynomial p such that $|p(b)| > \|p\|_V$. By the extension property, there is a function $\phi \in H^\infty(\Omega)$ such that $\phi|_V = p$ and $\|\phi\|_\Omega = \|p\|_V$. Let $\psi_b = \phi - p$. Then ψ_b vanishes on V and is nonzero on b. Therefore $V = \cap_{b \in \Omega \setminus V} Z_{\psi_b}$.

Locally, at any point a in V, the ring of germs of holomorphic functions is Noetherian [14, Thm. B.10]. Therefore V is locally the intersection of finitely many zero sets of functions in $H^\infty(\Omega)$, and it therefore is a holomorphic subvariety. □

In the next proof, we shall write X_* to mean $X \setminus \{0\}$.

Lemma 4.4. *Let V be a totally geodesic holomorphic subvariety of a strongly linearly convex domain Ω, and assume that V has the extension property. Then V is a complex manifold.*

Proof. Set $m = \dim_{\mathbb{C}} V$, and take $a \in V$. We want to show that V is a complex submanifold near a. If $m = 0$, there is nothing to prove, so we shall assume that $m > 0$. Take any point in $V \setminus \{a\}$ and a complex geodesic passing through it and a. According to Lemma 4.3, this geodesic is contained in V. Of course, it is a one-dimensional complex submanifold, and so trivially a two-dimensional real submanifold.

Claim.

(i) If $\mathcal{F}$ is a local $2k$-dimensional real submanifold smoothly embedded in V in a neighborhood of $a \in \mathcal{F}$ and there is a geodesic $f : \mathbb{D} \to V$ such that $f(0) = a$ and $f'(0) \notin \omega T_a \mathcal{F}$ for any $\omega \in \mathbb{T}$, then there is a local real submanifold of dimension $2k+2$ that is contained in V and that contains a.

(ii) Moreover, one can always find such a geodesic f whenever $k < m$ or if $k = m$ and V does not coincide with $\mathcal{F}$ near a.

To prove part (i) of the claim, let us take a geodesic f such that $f(0) = a$, and $f'(0) \notin \omega T_a\mathcal{F}$ for any $\omega \in \mathbb{T}$. Take any $t_0 \in (0,1)$, and set $a_0 = f(t_0)$. It follows from Lempert's theorem [20, 21] (see [18] for an exposition) that there are a neighborhood U of a and a smooth mapping $\Phi : U \times \mathbb{D} \to \mathbb{C}^d$ such that, for any z near a, a disk $\Phi(z,\cdot)$ is a geodesic passing through a_0 and z, and such that $\Phi(z,0) = a_0$ and $\Phi(z,t_z) = z$ for some $t_z > 0$. It also follows from Lempert's theory that $z \mapsto t_z$ is smooth. Observe that $\Phi(a,\lambda) = f(m_{t_0}(\lambda))$. Note that the mapping

$$\mathcal{F} \times \mathbb{D} \ni (z,\lambda) \mapsto \Phi(z,\lambda)$$

sends (a,t_0) to a. We shall show that its Jacobian is nondegenerate in a neighborhood of (a,t_0). This will imply that the image of this mapping is a smooth $(2k+2)$-dimensional real submanifold, thus proving part (i) of the claim.

Let $p : (-1,1)^{2k} \to \mathcal{F}$ give local coordinates for $\mathcal{F}$, $p(0) = a$. We need to compute the Jacobian of $(s,\lambda) \mapsto \Psi(s,\lambda) := \Phi(p(s),\lambda)$ at $(0,t_0)$. Write λ in coordinates $(x,y) \in \mathbb{R}^2$ and $\Psi = (\Psi_1,\dots,\Psi_{2d})$.

The Jacobian matrix of Ψ is the $(2k+2)$-by-$2d$ matrix with columns

$$\begin{pmatrix} \frac{\partial \Psi_i}{\partial s_j} \\ \\ \frac{\partial \Psi_i}{\partial x} \\ \\ \frac{\partial \Psi_i}{\partial y} \end{pmatrix}.$$

Differentiating $\Psi(s,r(s)) = p(s)$, where $r(s) = t_{p(s)}$, we get

$$\frac{\partial \Psi_i}{\partial s_j} + \frac{\partial \Psi_i}{\partial x}\frac{\partial r}{\partial s_j} \;=\; \frac{\partial p_i}{\partial s_j}.$$

So the rank of the Jacobian of Ψ is the same as the rank of the matrix with columns

$$\begin{pmatrix} \frac{\partial p_i}{\partial s_j} \\ \\ \frac{\partial \Psi_i}{\partial x} \\ \\ \frac{\partial \Psi_i}{\partial y} \end{pmatrix}. \tag{4.5}$$

Since $\Psi(0,\lambda) = f(m_{t_0}(\lambda))$, by differentiating with respect to λ, we find that $\partial\Psi/\partial\lambda(0,t_0) = -f'(0)/(1-t_0^2)$. So if $f'(0) \notin \omega T_a\mathcal{F}$ for any unimodular ω, the rank of (4.5) is two more than the rank of $(\frac{\partial p_i}{\partial s_j})$, which is $2k$. Therefore the Jacobian of Ψ is of rank $2k+2$, so we have established part (i) of the claim.

Proof of part (ii) of the claim, the existence of the geodesic, follows.

For $z \in V \setminus \{a\}$, let f_z denote a complex geodesic such that $f_z(0) = a$ and $f_z(t_z) = z$ for some $t_z > 0$.

Case: $k < m$. Note that the real dimension of the set $\{tf_z'(0) :\ z \in V, t > 0\}$ is equal at least to $2m$. To see this, one can proceed as follows: take $z_0 \in V_{reg} \setminus \{a\}$ such that $\dim_{z_0} V = m$. Let W be an $(2m-1)$-real dimensional manifold near z_0 that is contained in V and is transversal at this point to $(0,1) \ni r \mapsto f_{z_0}(r)$. Then the mapping $(0,\infty) \times W \to \mathbb{C}^m$ given by $(t,z) \mapsto tf_z'(0)$ is injective for z close to z_0, so its image is $2m$ real dimensional.

On the other hand, the real dimension of the set

$$\mathbb{T} \cdot T_a\mathcal{F} \ := \ \{\omega X : \ \omega \in \mathbb{T}, X \in T_a\mathcal{F}\}$$

is equal to $2k+1$ or $2k$. So the existence of the geodesic follows.

Case: $k = m$. Then $\mathcal{F}$ is a real submanifold of dimension $2m$ that is contained in an analytic set of complex dimension m. The singular points of V are a subset of complex dimension at most $m-1$. Looking at the regular points of V, we get that $\mathcal{F}$ is a totally complex manifold, which means that its tangent space has a complex structure, on the regular points. Since $\mathcal{F}$ is a submanifold near z, its tangent space depends continuously on z, so $\mathcal{F}$ has a complex structure on its tangent space at all points.

By [13, Lemma I.7.15], it follows that $\mathcal{F}$ must be a complex submanifold. In particular, its tangent space is invariant under complex multiplication.

Change variables so that $\mathcal{F}$ is the graph of a holomorphic mapping $\{(z', h(z')) : ||z' - a'|| < 2\epsilon_0\}$ near $a = (a', a'')$. Denote by S_ϵ the set

$$S_\epsilon \ = \ \{(z', h(z')) : \ ||z' - a'|| = \epsilon\}.$$

If there are $\epsilon > 0$ and $z' \in S_\epsilon$ such that $f'_{(z', h(z'))}(0) \notin T_a\mathcal{F}$, we are done, as $f_{(z', h(z'))}$ is a geodesic we are looking for. Otherwise, for ϵ small enough, say, $0 < \epsilon < \epsilon_1$, consider the mapping

$$\begin{aligned} \alpha_\epsilon : (0,\infty) \times S_\epsilon \quad &\to \quad (T_a\mathcal{F})_*, \\ (r, z) \quad &\mapsto \quad r f'_z(0). \end{aligned} \tag{4.6}$$

It follows from the uniqueness of geodesics (with the normalization that 0 maps to a and $f_z^{-1}(z)$ is positive) that α_ϵ is injective. By the invariance of domain theorem, α_ϵ is open. Therefore, as the range of the mapping α_ϵ is open and closed, α_ϵ is surjective.

Take a point $w \in V \setminus \mathcal{F}$ close to a. Let g be a complex geodesic for a and w such that $g(0) = a$ and $g(t_w) = w$. To finish the proof of the claim, it suffices to show that $g'(0) \notin T_a\mathcal{F}$. Seeking a contradiction, suppose that $g'(0)$ lies in $T_a\mathcal{F}$.

By the surjectivity of α_ϵ, we get that there is $(r, z) \in (0,\infty) \times S_\epsilon$ such that $r f'_z(0) = g'(0)$. The uniqueness of geodesics implies that $f_z = g$ and consequently $g(t_z) = f_z(t_z) = z$.

Let

$$H(z', z'') \ := \ (||z' - a'||^2, ||z'' - h(z')||^2).$$

Then $H \circ g$ is a real analytic mapping, which is well defined on a real interval surrounding 0. Observe that $H \circ g(t_z) = (\epsilon^2, 0)$ since z is in S_ϵ. Applying this formula for any δ in the interval $(0, \epsilon_1)$, we get that there are $t(\delta)$ in $(0,1)$ so that $H \circ g(t(\delta)) = (\delta^2, 0)$. It is an immediate consequence of this equality that $t(\delta) \neq t(\delta')$ if $\delta \neq \delta'$. It follows from the identity principle that the second component of $t \mapsto H \circ g(t)$ is identically equal to 0. In particular, it also vanishes at t_w, so w is in $\mathcal{F}$, a contradiction.

Having proved the claim, we finish the proof of the lemma in the following way. Let $\mathcal{F}_1$ be a complex geodesic contained in V that passes through a. If $m > 1$ for $1 \leq k \leq m-1$, by part (ii) of the claim, we can find a geodesic f through a such that $f'(0)$ is not in $\omega T_a\mathcal{F}_k$ for any complex ω, so by part (i) we can find a real $(2k+2)$-dimensional smoothly embedded manifold $\mathcal{F}_{k+1}$ contained in V and

containing a. If $\mathcal{F}_m$ does not equal V near a, then by part (ii), we could find a $\mathcal{F}_{m+1}$ contained in V, which is ruled out by the dimension count. Therefore $\mathcal{F}_m = V$ near a, so V is a smooth real submanifold, and hence, as already shown, a complex submanifold. □

Combining Lemmas 4.2 and 4.4, we finish the proof of Theorem 1.6.

Proof of Corollary 1.7. If $d = 2$, then we have shown that if V has the extension property and is not a singleton or all of Ω, then it is a one-dimensional totally geodesic set. So there exists a Kobayashi extremal $f : \mathbb{D} \to \Omega$ whose range is exactly V. By Lempert's theorem, there is a left inverse $L : \Omega \to \mathbb{D}$. Let $r = f \circ L$; this is a retract from Ω onto V. □

5. Spectral sets

Let Ω be a bounded open set in $\mathbb{C}^d$, and let $V \subseteq \Omega$; they shall remain fixed for the remainder of this section. Let $A(\Omega)$ denote the algebra of holomorphic functions on Ω that extend to be continuous on the closure $\overline{\Omega}$, equipped with the supremum norm. For any positive finite measure μ supported on $\overline{\Omega}$, let $A^2(\mu)$ denote the closure of $A(\Omega)$ in $L^2(\mu)$.

A point $\lambda \in \Omega$ is called a bounded point evaluation of $A^2(\mu)$ if there exists a constant C so that

$$|f(\lambda)| \ \leq \ C\|f\| \qquad \forall f \in A(\Omega). \tag{5.1}$$

If (5.1) holds, then by the Riesz representation theorem, there is a function $k_\lambda^\mu \in A^2(\mu)$ such that

$$f(\lambda) \ = \ \langle f, k_\lambda^\mu \rangle \qquad \forall f \in A(\Omega). \tag{5.2}$$

Given a set $\Lambda \subseteq \Omega$, we say the measure μ is dominating for Λ if every point of Λ is a bounded point evaluation for $A^2(\mu)$. We shall need the following theorem of Cole, Lewis, and Wermer [12]; similar results were proved by Amar [6] and Nakazi [22]. See [2, Thm. 13.36] for an exposition. For the polydisk or the ball, one can impose extra restrictions on the measures μ that need to be checked [7, 27].

Theorem 5.3. *Let $\{\lambda_1, \ldots, \lambda_N\} \subseteq \Omega$ and $\{w_1, \ldots, w_N\} \subseteq \mathbb{C}$ be given. For every $\epsilon > 0$, there exists a function $f \in A(\Omega)$ of norm at most $1 + \varepsilon$ that satisfies*

$$f(\lambda_i) \ = \ w_i \qquad \text{for } i = 1, \ldots, N$$

if and only if, for every measure μ supported on $\partial\Omega$ that dominates $\{\lambda_1, \ldots, \lambda_N\}$, we have

$$\Big[(1 - w_i\overline{w_j})\langle k_{\lambda_j}^\mu, k_{\lambda_i}^\mu \rangle_{A^2(\mu)}\Big]_{i,j=1}^N \ \geq \ 0. \tag{5.4}$$

Proof of Theorem 1.12. We shall actually show slightly more: for any function $f \in \mathcal{A}$ (which for this theorem is the algebra of polynomials), we shall show that f can be extended to a function ϕ in $H^\infty(\Omega)$ of the same norm that agrees with f on V if and only if

$$\|f(T)\| \ \leq \ \sup_{\lambda \in V} |f(\lambda)| \qquad \forall T \text{ subordinate to } V. \tag{5.5}$$

One direction is easy. Suppose that V has the $\mathcal{A}$ extension property, and suppose that T is subordinate to V. By the extension property, there exists $\phi \in H^\infty(\Omega)$

such that $\phi|_V = f|_V$ and $\|\phi\| = \sup_V |f|$. Since T is subordinate to V and $f - \phi$ vanishes on V, we get that $f(T) = \phi(T)$, so

$$\|f(T)\| \;=\; \|\phi(T)\| \;\leq\; \|\phi\| \;=\; \sup_{\lambda \in V} |f(\lambda)|,$$

where the inequality comes from the fact that Ω is a spectral set for T.

The converse direction is more subtle. Suppose $\sup_{\lambda \in V} |f(\lambda)| = 1$, and assume that we cannot extend f to a function ϕ of norm 1 in $H^\infty(\Omega)$. We shall construct T subordinate to V so that (5.5) fails.

Let $\{\lambda_j\}$ be a countable dense set in V. Let $w_j = f(\lambda_j)$. For each N, let $E_N = \{\lambda_1, \ldots, \lambda_N\}$. If, for each N, one could find $\phi_N \in A(\Omega)$ of norm at most $1 + \frac{1}{N}$ and that satisfies

$$\phi_N(\lambda_i) \;=\; f(\lambda_i), \qquad 1 \leq i \leq N,$$

then by Montel's theorem, some subsequence of (ϕ_N) would converge to a function ϕ in the closed unit ball of $H^\infty(\Omega)$ that agreed with f on a dense subset of V, and hence on all of V. So by Theorem 5.3, there must be some N, and some measure μ that dominates E_N, so that (5.4) fails. Fix such an N and such a μ.

Let $k_j = k^\mu_{\lambda_j}$, and let $\mathcal{M}$ be the linear span of $\{k_1, \ldots, k_N\}$. Define a d-tuple of operators $T = (T_1, \ldots, T_d)$ on $\mathcal{M}$:

$$T_r^* \; k_j \;=\; \overline{\lambda_j^r} \; k_j, \qquad 1 \leq r \leq d, \; 1 \leq j \leq N. \tag{5.6}$$

(We write λ_j^r for the rth component of λ_j). By (5.6), the spectrum of T^* is $\{\overline{\lambda_1}, \ldots, \overline{\lambda_N}\}$, so the spectrum of T is $E_N \subseteq V$.

If $\psi \in H^\infty(\Omega)$, let $\psi^\cup(z) = \overline{\psi(\overline{z})}$. From (5.6), it follows that

$$\psi(T)^* k_j \;=\; \psi^\cup(T^*) k_j \;=\; \overline{\psi(\lambda_j)} k_j. \tag{5.7}$$

Indeed, since T is a d-tuple of matrices, $\psi(T)$ depends only on the value of ψ and a finite number of derivatives on the spectrum of T; so one can approximate ψ by a polynomial, and for polynomials, (5.7) is immediate.

We have that T is the compression to $\mathcal{M}$ of multiplication by the coordinate functions on $A^2(\mu)$. Let P be orthogonal projection from $A^2(\mu)$ onto $\mathcal{M}$. If $g \in A(\Omega)$, then

$$\|g(T)\| \;=\; \|P M_g P\| \;\leq\; \|g\|_{A(\Omega)},$$

where M_g is multiplication by g in $A^2(\mu)$. So T has Ω as a spectral set. By (5.7), if ψ vanishes on V, then $\psi(T) = 0$, so T is subordinate to V.

We want to show that $\|f(T)\| > 1$. If it were not, then

$$I - f(T) f(T)^* \;\geq\; 0. \tag{5.8}$$

Evaluating the left-hand side of (5.8) on $v = \sum a_j k_j$ and using (5.7), one gets

$$\langle (I - f(T) f(T)^*) v, v \rangle \;=\; \sum_{i,j=1}^{N} \overline{a_i} a_j (1 - w_i \overline{w_j}) \langle k_j, k_i \rangle. \tag{5.9}$$

But we chose N and μ so that for some choice of a_j, (5.9) is negative. This contradicts (5.8). $\square$

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