Facial behaviour of analytic functions on the bidisk

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Abstract

We prove that if φ is an analytic function bounded by 1 on the bidisk $\mathbb{D}^2$ and τ is a point in a face of $\mathbb{D}^2$ at which φ satisfies Carathéodory's condition then both φ and the angular gradient $\nabla\varphi$ exist and are constant on the face. Moreover, the class of all φ with prescribed $\varphi(\tau)$ and $\nabla\varphi(\tau)$ can be parametrized in terms of a function in the two-variable Pick class. As an application we solve an interpolation problem with nodes lying on faces of the bidisk.

1 Introduction

We study functions in the Schur class $\mathcal{S}_2$ of the bidisk $\mathbb{D}^2$, that is, functions analytic and bounded by 1 on $\mathbb{D}^2$, and in particular their behaviour on faces of $\mathbb{D}^2$. A *face* of $\mathbb{D}^2$ is a subset of the topological boundary $\partial(\mathbb{D}^2)$ of $\mathbb{D}^2$ having one of the forms $\{\tau^1\} \times \mathbb{D}$ or $\mathbb{D} \times \{\tau^2\}$, where $|\tau^1| = |\tau^2| = 1$ and $\mathbb{D}$ denotes the open unit disk in the complex plane $\mathbb{C}$. Of course a function $\varphi \in \mathcal{S}_2$ need not have values at all points of $\partial(\mathbb{D}^2)$, but it is known [7, 1] that φ has a nontangential limit at any point $\tau \in \partial(\mathbb{D}^2)$ for which the Carathéodory condition

$$\liminf_{\lambda \to \tau} \frac{1 - |\varphi(\lambda)|}{1 - \|\lambda\|} < \infty \tag{1.1}$$

holds, where

$$\|(\lambda^1, \lambda^2)\| = \max\{|\lambda^1|, |\lambda^2|\}.$$

We shall say that $\tau \in \partial(\mathbb{D}^2)$ is a *B-point for* φ if condition (1.1) holds. We denote the nontangential limit (explained in detail below) of φ at a B-point τ by $\varphi(\tau)$.

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The purpose of this paper is to show that if $\varphi \in \mathcal{S}_2$ has a B-point τ lying on a face of $\mathbb{D}^2$ then strong consequences follow: both φ and the angular gradient $\nabla\varphi$ exist and are constant on the face, and in fact

$$\nabla\varphi(\sigma) = \varphi(\tau)\overline{\tau^1}\begin{pmatrix}\alpha\\0\end{pmatrix}$$

for all $\sigma \in \{\tau^1\} \times \mathbb{D}$, where α is the value of the lim inf on the left hand side of inequality (1.1). Moreover, for any given $\tau \in \mathbb{T}\times\mathbb{D}$, the functions in $\varphi \in \mathcal{S}_2$ having a B-point at τ and prescribed values of $\varphi(\tau)$ and $\nabla\varphi(\tau)$ can be parametrized in terms of a function in the two-variable Pick class.

In Section 2 we establish notation and introduce the notion of angular gradient for functions in the two-variable Schur and Pick classes. We also explain the notion of a *model* of a function in the Schur class and recall one property of models. In Section 3 we prove the constancy result mentioned above (Theorem 3.2). In Section 4 we present the precise relationship between functions in the Schur and Pick classes given by the Cayley transform and thereby obtain an analog of Theorem 3.2 for the Pick class. In Section 5 we present Julia's reduction method for the Pick class, and in Section 6 we derive two parametrization results, Theorems 6.1 and 6.6. In Section 7 we apply our parametrization theorem to solve an interpolation problem.

2 Notation and definitions

We denote by Π the upper half-plane $\{z \in \mathbb{C} : \text{Im } z > 0\}$. The one- and two-variable *Pick classes*, denoted by $\mathcal{P}$ and $\mathcal{P}_2$, are the sets of analytic functions on Π and Π^2 respectively with non-negative imaginary part. The one- and two-variable *Schur classes* $\mathcal{S}$ and $\mathcal{S}_2$ are the the sets of analytic functions on $\mathbb{D}$ and $\mathbb{D}^2$ respectively that are bounded by 1 in absolute value.

We shall need the notion of *nontangential approach*. For a domain $\Omega \subset \mathbb{C}^n$, a set $S \subset \Omega$ is said to approach $\tau \in \partial\Omega$ nontangentially if $\tau \in S^-$ and

$$\frac{\|\lambda - \tau\|}{\text{dist}(\lambda, \mathbb{C}^n \setminus \Omega)} \text{ is bounded for } \lambda \in S.$$

The smallest $c \geq 1$ that bounds the above set is called the *aperture* of S.

In one variable the Julia-Carathéodory Theorem [5, 6, 10] tells us that a function $\varphi \in \mathcal{S}$ has an angular derivative at any B-point τ. More fully, if the Carathéodory condition holds (replace $\|\lambda\|$ by $|\lambda|$ in condition (1.1)), then

(1) $\varphi(\lambda)$ tends to a limit $\varphi(\tau)$ as $\lambda \to \tau$ in any set that approaches τ nontangentially;

(2) the difference quotient $(\varphi(\lambda)-\varphi(\tau))/(\lambda-\tau)$ tends to a limit $\varphi'(\tau)$ as $\lambda \to \tau$ in any set that approaches τ nontangentially;

(3) $\varphi'(\lambda) \to \varphi'(\tau)$ as $\lambda \to \tau$ nontangentially;

(4) the limit inferior in relation (1.1) is equal to the lim inf along the radius $\lambda = r\tau,\ 0 \le r < 1$, and is in fact a limit as $r \to 1$.

As we have mentioned, (1) remains true for the polydisk [7, 1], and so $\varphi \in \mathcal{S}_2$ has a value $\varphi(\tau) \in \mathbb{T}$ at any B-point τ. However, the analog of (2) is false in two variables: in general $\varphi \in \mathcal{S}_2$ does not have an angular gradient at all its B-points (see for example Remark 3.14 below). We are led to study those points τ at which φ does have an angular gradient.

Definition 2.1. *Let $\varphi \in \mathcal{S}_2, \tau \in \partial(\mathbb{D}^2)$.*
(1) *For $S \subseteq \mathbb{D}^2$, $\tau \in S^-$ we say that φ* has a holomorphic differential at τ on S *if there exist $\omega, \eta^1, \eta^2 \in \mathbb{C}$ such that, for all $\lambda \in S$,*

$$\varphi(\lambda) = \omega + \eta^1(\lambda^1 - \tau^1) + \eta^2(\lambda^2 - \tau^2) + e(\lambda) \tag{2.2}$$

where

$$\lim_{\lambda \to \tau,\ \lambda \in S} \frac{e(\lambda)}{||\lambda - \tau||} = 0. \tag{2.3}$$

(2) *We say that τ is a C-*point for φ *if, for every set S that approaches τ nontangentially, φ has a holomorphic differential on S and ω in relation* (2.2) *has modulus* 1.
(3) *If τ is a C-point for φ we define the* angular gradient $\nabla\varphi(\tau)$ *of φ at τ to be the vector $\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix}$, where φ has holomorphic differential* (2.2) *on some set that approaches τ nontangentially.*

It is clear that, when τ is a C-point for φ, the quantities ω, η^1, η^2 in equation (2.2) are the same for every nontangential approach region S, and so the definition of $\nabla\varphi(\tau)$ in (3) is unambiguous.

An apparent drawback of the above definition of C-point is that a condition must hold for *every* set S that approaches τ nontangentially. However, we showed in [3, Remark 8.12] that the condition need only be checked for a single suitable set S.

Analogous definitions of C-points and angular gradients apply to functions in the Pick class $\mathcal{P}_2$.

Our approach makes use of the notion of a model of a Schur-class function, as developed in [3].

Definition 2.4. *Let $\varphi \in \mathcal{S}$. We say that $(\mathcal{M}, u)$ is a* model of φ *if $\mathcal{M} = \mathcal{M}^1 \oplus \mathcal{M}^2$ is an orthogonally decomposed separable Hilbert space and $u : \mathbb{D}^2 \to \mathcal{M}$ is an analytic map such that, for all $\lambda, \mu \in \mathbb{D}^2$,*

$$1 - \overline{\varphi(\mu)}\varphi(\lambda) = (1 - \overline{\mu^1}\lambda^1)\langle u^1_\lambda, u^1_\mu\rangle + (1 - \overline{\mu^2}\lambda^2)\langle u^2_\lambda, u^2_\mu\rangle. \tag{2.5}$$

In equation (2.5) we have written u_λ for $u(\lambda)$, $u^1_\lambda = P_{\mathcal{M}^1}u_\lambda$, and $u^2_\lambda = P_{\mathcal{M}^2}u_\lambda$. More generally, if $v \in \mathcal{M}$, we set $v^1 = P_{\mathcal{M}^1}v$ and $v^2 = P_{\mathcal{M}^2}v$. If $\lambda \in \mathbb{D}^2$, we may regard λ as an operator on $\mathcal{M}$ by letting

$$\lambda v = \lambda^1 v^1 + \lambda^2 v^2$$

for $v \in \mathcal{M}$.

Our methods depend on the fact that every function in $\mathcal{S}_2$ has a model in the sense of Definition 2.4, as was proved in [2].

If $(\mathcal{M}, u)$ is a model of $\varphi \in \mathcal{S}_2$ then we define the *nontangential cluster set* X_τ of the model at a B-point τ of φ to be the set of weak limits of weakly convergent sequences u_{λ_n} over all sequences λ_n that converge nontangentially to τ in $\mathbb{D}$. Here is a simple but powerful consequence of the model relation (2.5).

Proposition 2.6. *Let $\tau \in \mathbb{T} \times \mathbb{D}$ be a B-point for $\varphi \in \mathcal{S}_2$ and let $(\mathcal{M}, u)$ be a model of φ. Then*

(1) $\emptyset \neq X_\tau \subset \mathcal{M}^1$;

(2) *there exists $\omega \in \mathbb{T}$ such that, for every $x = x^1 \in X_\tau$ and $\lambda \in \mathbb{D}^2$,*

$$1 - \overline{\omega}\varphi(\lambda) = (1 - \overline{\tau^1}\lambda)\langle u^1_\lambda, x^1\rangle. \tag{2.7}$$

A detailed proof is given in [3, Proposition 4.2]. Nonemptiness of X_τ follows from the fact that u_{λ_n} is bounded for any sequence (λ_n) in $\mathbb{D}^2$ that converges to τ nontangentially, while the relation (2.7) is derived by substituting $\mu = \mu_n$ in equation (2.5) for a suitable sequence (μ_n) converging to τ and then passing to a limit.

It is shown in [3, Corollary 8.11] that τ is a C-point for φ if and only if X_τ is a singleton set; we denote the unique member of this set by u_τ.

3 Facial B-points

A function $\varphi \in \mathcal{S}_2$ can have a B-point in $\partial(\mathbb{D}^2) \setminus \mathbb{T}^2$, say at $\tau \in \mathbb{T} \times \mathbb{D}$; we shall call such a τ a *facial B-point* for φ. Facial B-points can arise in a trivial way, when φ is independent of λ^2, but can also occur non-trivially. Consider for example the rational inner function

$$\varphi(\lambda) = \frac{1 + \lambda^1 + \lambda^2 - 3\lambda^1\lambda^2}{3 - \lambda^1 - \lambda^2 - \lambda^1\lambda^2}. \tag{3.1}$$

φ is analytic and equal to 1 at every point of the face $\{1\} \times \mathbb{D}$ of the bidisk, and so every point of this face is a C-point for φ. Likewise $\varphi = 1$ at every point of the face $\mathbb{D} \times \{1\}$. The example illustrates a general phenomenon. We shall denote by Δ the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

Theorem 3.2. *Let $\tau \in \mathbb{T} \times \mathbb{D}$ be a B-point for $\varphi \in \mathcal{S}_2$. Then*

(1) *every point of $\{\tau^1\} \times \Delta$ is a B-point for φ;*

(2) *every point of $\{\tau^1\} \times \mathbb{D}$ is a C-point for φ;*

(3) *φ is constant on $\{\tau^1\} \times \Delta$;*

(4) $\nabla\varphi$ *is constant on* $\{\tau^1\}\times\mathbb{D}$*, with value*

$$\nabla\varphi(\sigma)=\varphi(\tau)\overline{\tau^1}\alpha\begin{pmatrix}1\\0\end{pmatrix}\qquad \textit{for every } \sigma\in\{\tau^1\}\times\mathbb{D} \tag{3.3}$$

where

$$\alpha=\liminf_{\lambda\to\tau}\frac{1-|\varphi(\lambda)|}{1-\|\lambda\|}.$$

Proof. Let $(\mathcal{M},u)$ be a model of φ. By Proposition 2.6, since $|\tau^2|<1$, there exists $x\in\mathcal{M}$ and $\omega\in\mathbb{T}$ such that $x^2=0$ and

$$1-\bar{\omega}\varphi(\lambda)=(1-\overline{\tau^1}\lambda^1)\langle u^1_\lambda,x^1\rangle \tag{3.4}$$

and so

$$1-|\varphi(\lambda)|\le|1-\overline{\omega}\varphi(\lambda)|\le|1-\overline{\tau^1}\lambda^1|\ |\langle u^1_\lambda,x^1\rangle| \tag{3.5}$$

for all $\lambda\in\mathbb{D}^2$.
(1) Consider any point $\sigma=(\tau^1,\sigma^2)\in\{\tau^1\}\times\mathbb{D}$. Let

$$S_\sigma=\{((1-t)\tau^1,\sigma^2):0<t\le 1-|\sigma^2|\}.$$

S_σ approaches σ nontangentially with aperture 1, and we have

$$|1-\overline{\tau^1}\lambda^1|=1-\|\lambda\| \tag{3.6}$$

for $\lambda\in S_\sigma$. By [3, Remark 5.6], for $\lambda\in S_\sigma$,

$$||u_\lambda||\le 2||x^1||, \tag{3.7}$$

which, together with (3.5) and (3.6), implies that

$$\frac{1-|\varphi(\lambda)|}{1-||\lambda||}=\frac{1-|\varphi(\lambda)|}{|1-\overline{\tau^1}\lambda^1|}\le 2||x^1||^2 \tag{3.8}$$

for all $\lambda\in S_\sigma$. Hence σ is a B-point for φ.

Now consider any point $\rho\in\{\tau^1\}\times\mathbb{T}$. For any $r\in(0,1)$ observe that

$$S_{(\tau^1,(1-r)\rho^2)}=\{((1-t)\tau^1,(1-r)\rho^2):0<t\le r\},$$

and so in particular $(1-t)\rho\in S_{(\tau^1,(1-t)\rho^2)}$. The bound (3.7) is valid for all $\lambda\in\cup_{\sigma\in\{\tau^1\}\times\mathbb{D}}S_\sigma$ and hence for $\lambda=(1-t)\rho$. We therefore have the bound (3.8) for $\lambda=(1-t)\rho$ which tends to ρ as $t\to 0+$. Hence ρ is a B-point for φ.

(2) According to [3, Lemma 8.10], the nontangential cluster set of u at any facial B-point σ comprises a single vector in $\mathcal{M}$, and so every facial B-point is a C-point for φ.

(3) Equation (3.4) can be re-written

$$\varphi(\lambda) = \omega + \omega\overline{\tau^1}(\lambda^1 - \tau^1)\langle u^1_\lambda, x^1\rangle.$$

Consider $\sigma \in \{\tau^1\} \times \mathbb{D}$. Substitute λ_n for λ and let $\lambda_n \to \sigma$ in S_σ; by inequality (3.7), the second term on the right hand side tends to zero, and we find that

$$\varphi(\sigma) \stackrel{\text{def}}{=} \lim_{\lambda \stackrel{\text{nt}}{\to} \sigma} \varphi(\lambda) = \omega.$$

For $\rho \in \{\tau^1\} \times \mathbb{T}$ we may substitute $\lambda = \lambda_n = (1 - t_n)\rho$ where $t_n \to 0+$, and similar reasoning applies. Thus φ is constant on $\{\tau^1\} \times \Delta$.

(4) We prove that $\nabla\varphi$ is also constant. For any $\sigma = (\tau^1, \zeta),\ \zeta \in \mathbb{D}$, let

$$\nabla\varphi(\sigma) = \begin{pmatrix} \eta^1(\zeta) \\ \eta^2(\zeta) \end{pmatrix}.$$

By [3, Corollary 8.13] and the fact that $u^2_\sigma = 0$ we have

$$\begin{pmatrix} \eta^1(\zeta) \\ \eta^2(\zeta) \end{pmatrix} = \nabla\varphi(\sigma) = \varphi(\sigma) \begin{pmatrix} \overline{\sigma^1}||u^1_\sigma||^2 \\ \overline{\sigma^2}||u^2_\sigma||^2 \end{pmatrix} = \varphi(\tau)\overline{\tau^1} \begin{pmatrix} \|u^1_\sigma\|^2 \\ 0 \end{pmatrix}, \tag{3.9}$$

and hence

$$\overline{\varphi(\tau)}\tau^1\eta^1(\zeta) \geq 0 \text{ for all } \zeta \in \mathbb{D}. \tag{3.10}$$

The points $(\tau^1 - t\tau^1, \zeta),\ 0 < t \leq 1$, approach σ nontangentially, and so we have

$$\eta^1(\zeta) = -\lim_{t \to 0+} \overline{\tau^1}\frac{\varphi(\tau^1 - t\tau^1, \zeta) - \varphi(\tau)}{t} \stackrel{\text{say}}{=} \lim_{t \to 0+} F_t(\zeta).$$

We claim that the functions $F_t,\ 0 < t \leq 1$, are uniformly bounded on $\mathbb{D}$. Indeed, by equation (3.4),

$$\varphi(\tau^1 - t\tau^1, \zeta) - \varphi(\tau) = -\varphi(\tau)t\langle u_{(\tau^1 - t\tau^1, \zeta)}, x^1\rangle$$

and therefore

$$|F_t(\zeta)| \leq \|u_{(\tau^1 - t\tau^1, \zeta)}\|\ \|x^1\|.$$

The set $\{(\tau^1 - t\tau^1, \zeta) : 0 < t \leq 1\}$ approaches σ with aperture 1, and so, by [3, Remark 5.6],

$$\|u_{(\tau^1 - t\tau^1, \zeta)}\| \leq 2\|x^1\| \tag{3.11}$$

and consequently $|F_t(\zeta)| \leq 2\|x^1\|^2$ for $0 < t \leq 1,\ \zeta \in \mathbb{D}$. Since the F_t are analytic in $\mathbb{D}$, so is their pointwise limit η^1. In view of the positivity relation (3.10), it follows that η^1 is constant on $\mathbb{D}$. By equation (3.9) and [3, Theorem 5.10] we therefore have

$$\|u^1_\sigma\|^2 = \|u^1_\tau\|^2 = \|u_\tau\|^2 = \alpha, \tag{3.12}$$

and so the constant value of $\nabla\varphi(\sigma)$ is given by equation (3.3). □

Corollary 3.13. *If a function $\varphi \in \mathcal{S}_2$ has a B-point in $\mathbb{T} \times \mathbb{D}$ and another in $\mathbb{D} \times \mathbb{T}$ then φ takes the same value at the two B-points.*

For if $\tau \in \mathbb{T} \times \mathbb{D}$ and $\sigma \in \mathbb{D} \times \mathbb{T}$ are B-points then the constant value of φ on both of the closed faces must equal $\varphi(\tau^1, \sigma^2)$.

Remark 3.14. It is not the case that if φ has a facial B-point $\tau \in \mathbb{T} \times \mathbb{D}$ then every point of the *closed* face $\{\tau^1\} \times \Delta$ is necessarily a C-point for φ. A counterexample is furnished by the rational inner function

$$\psi(\lambda) = \frac{2\lambda^1\lambda^2 - \lambda^1 - \lambda^2}{2 - \lambda^1 - \lambda^2}.$$

The face $\{1\} \times \mathbb{D}$ consists of B-points for ψ, but the point $(1, 1)$ is not a C-point for ψ, and so $\nabla\psi(1, 1)$ is undefined. This example is analysed in [3, Section 6].

A modification of the proof of Theorem 3.2 yields a slightly stronger result.

Proposition 3.15. *Let $\tau \in \mathbb{T} \times \mathbb{D}$ be a B-point for $\varphi \in \mathcal{S}_2$. For every model $(\mathcal{M}, u)$ of φ, for all $\sigma \in \{\tau^1\} \times \mathbb{D}$ we have $u_\sigma = u_\tau$.*

Proof. Choose $x \in \mathcal{M}$ as in the preceding proof. For $\zeta \in \mathbb{D}$ let $\sigma = (\tau^1, \zeta)$. We claim that $u_{(\tau^1 - t\tau^1, \zeta)} \to u_\sigma$ as $t \to 0+$. Indeed, $u_{(\tau^1 - t\tau^1, \zeta)} = u_{\sigma - t\delta}$ where $\delta = (\tau^1, 0)$, and by [3, Theorem 7.1], $u_{\sigma - t\delta}$ tends to a limit in the cluster set X_σ of u at σ as $t \to 0+$. Since σ is a C-point for φ, X_σ comprises the unique vector u_σ, which proves the claim.

Consider the functions $\zeta \mapsto u_{(\tau^1 - t\tau^1, \zeta)} : \mathbb{D} \to \mathcal{M},\ 0 < t \leq 1$. It follows from inequality (3.11) that these analytic functions are uniformly bounded on $\mathbb{D}$. Since they are also pointwise convergent on $\mathbb{D}$ to the function $\zeta \mapsto u_{(\tau^1, \zeta)} = u_\sigma$, the latter map is analytic on $\mathbb{D}$. By equation (3.12), $\|u_\sigma\|$ is constant on $\mathbb{D}$. However, an analytic Hilbert-space-valued function f such that $\|f(.)\|$ is constant is itself a constant function. Thus u_σ is constant on $\{\tau^1\} \times \mathbb{D}$. □

4 The Cayley transform for $\mathcal{S}_2$ and $\mathcal{P}_2$

The Cayley transform

$$C : \mathbb{D} \to \Pi \text{ and } \mathbb{C} \setminus \{1\} \to \mathbb{C} \setminus \{-\mathrm{i}\} : \lambda \mapsto \mathrm{i}\frac{1 + \lambda}{1 - \lambda}$$

enables us to pass back and forth between $\mathcal{S}_2$ and $\mathcal{P}_2$. The transform C also acts (co-ordinatewise) from $\mathbb{D}^2$ to Π^2. The relationship between properties of a function $\varphi \in \mathcal{S}_2$ and those of the corresponding function $h \in \mathcal{P}_2$ is straightforward, but not totally trivial, and so we summarize it here.

The variables $\lambda \in \mathbb{D},\ z \in \Pi$ will be related by $z = C(\lambda)$, so that

$$z = \mathrm{i}\frac{1 + \lambda}{1 - \lambda}, \qquad \lambda = \frac{z - \mathrm{i}}{z + \mathrm{i}}. \tag{4.1}$$

Similarly, variables $\lambda \in \mathbb{D}^2$ and $z \in \Pi^2$ are related co-ordinatewise ($z^j = C(\lambda^j)$ etc).

Let $h \in \mathcal{P}_2$ correspond to $\varphi \in \mathcal{S}_2$, φ not identically equal to 1, under the Cayley transform, that is,

$$h(z) = \mathrm{i}\frac{1+\varphi(\lambda)}{1-\varphi(\lambda)}, \qquad \varphi(\lambda) = \frac{h(z)-\mathrm{i}}{h(z)+\mathrm{i}}, \tag{4.2}$$

where λ, z are related by equations (4.1). We consider a B-point $\tau \in \partial(\mathbb{D}^2)$ for φ, and we let $x \in \partial(\mathbb{R}^2)$ correspond to τ:

$$x^j = \mathrm{i}\frac{1+\tau^j}{1-\tau^j}, \qquad \tau^j = \frac{x^j-\mathrm{i}}{x^j+\mathrm{i}}, \qquad \text{for } j = 1, 2. \tag{4.3}$$

Let $\varphi(\tau) = \omega$, so that $|\omega| = 1$. Then $h(x) = \xi$ where $\xi \in \mathbb{R} \cup \{\infty\}$ is the transform of $\omega \in \mathbb{T}$:

$$\xi = \mathrm{i}\frac{1+\omega}{1-\omega}, \qquad \omega = \frac{\xi-\mathrm{i}}{\xi+\mathrm{i}}.$$

We ask: what conditions on h, x correspond to τ being a B-point or a C-point of φ, and what is the relation between $\nabla\varphi(\tau)$ and $\nabla h(x)$ in the case that τ is a C-point of φ?

Let us assume that τ^1, τ^2 and ω are all different from 1 (else we may compose with rotations of the circle). It is then the case that x^1, x^2 and ξ are all real numbers (not ∞). By a straightforward calculation,

$$\frac{1-|\varphi(\lambda)|^2}{1-\|\lambda\|^2} = \frac{\mathrm{Im}\ h(z)}{|h(z)+\mathrm{i}|^2}\max_j \frac{|z^j+\mathrm{i}|^2}{\mathrm{Im}\ z^j}.$$

Hence

$$\liminf_{\lambda\to\tau} \frac{1-|\varphi(\lambda)|^2}{1-\|\lambda\|^2} = \liminf_{z\to x} \frac{1}{|\xi+\mathrm{i}|^2}\max_j \frac{|x^j+\mathrm{i}|^2 \mathrm{Im}\ h(z)}{\mathrm{Im}\ z^j}. \tag{4.4}$$

It follows that τ is a B-point for φ if and only if

$$\liminf_{z\to x} \frac{\mathrm{Im}\ h(z)}{\mathrm{Im}\ z^j} < \infty \text{ for } j = 1, 2. \tag{4.5}$$

We shall say that $x \in \partial(\mathbb{R}^2)$ is a B-point for $h \in \mathcal{P}_2$ whenever the relation (4.5) holds.

Note that, for $\tau \in \mathbb{T}\times\mathbb{D}$, $x \in \mathbb{R}\times\Pi$, the limits inferior in equation (4.4) occur for $j = 1$. Hence, if

$$\alpha \stackrel{\text{def}}{=} \liminf_{\lambda\to\tau} \frac{1-|\varphi(\lambda)|}{1-\|\lambda\|} = \liminf_{\lambda\to\tau} \frac{1-|\varphi(\lambda)|}{1-|\lambda^1|} \tag{4.6}$$

and

$$\beta \stackrel{\text{def}}{=} \liminf_{z\to x} \max_j \frac{\mathrm{Im}\ h(z)}{\mathrm{Im}\ z^j} = \liminf_{z\to x} \frac{\mathrm{Im}\ h(z)}{\mathrm{Im}\ z^1} \tag{4.7}$$

then equation (4.4) yields

$$\alpha = \left| \frac{x^1 + \mathrm{i}}{\xi + \mathrm{i}} \right|^2 \beta. \tag{4.8}$$

Now consider a C-point τ for $\varphi \in \mathcal{S}_2$. There exist $\omega \in \mathbb{T}$ and $\eta^1, \eta^2 \in \mathbb{C}$ such that equation (2.2) holds on any set S that approaches τ nontangentially. Nontangential approach is preserved by the Cayley transform, and

$$\|\lambda - \tau\| \to 0 \Leftrightarrow \|z - x\| \to 0.$$

Let us rewrite equation (2.2) in terms of the variables z, x and h. We have

$$\begin{aligned} h(z) &= \mathrm{i}\frac{1 + \varphi(\lambda)}{1 - \varphi(\lambda)} \\ &= \mathrm{i}\frac{1 + \omega + \eta \cdot (\lambda - \tau) + o(\|\lambda - \tau\|)}{1 - \omega - \eta \cdot (\lambda - \tau) + o(\|\lambda - \tau\|)} \\ &= \xi\frac{1 + (1+\omega)^{-1}\eta \cdot (\lambda - \tau) + o(\|\lambda - \tau\|)}{1 - (1-\omega)^{-1}\eta \cdot (\lambda - \tau) + o(\|\lambda - \tau\|)} \\ &= \xi\left\{1 + \left(\frac{1}{1+\omega} + \frac{1}{1-\omega}\right)\eta \cdot (\lambda - \tau) + o(\|\lambda - \tau\|)\right\} \\ &= \xi + \frac{2\xi}{1-\omega^2}\eta \cdot (\lambda - \tau) + o(\|\lambda - \tau\|) \end{aligned} \tag{4.9}$$

as $\lambda \to \tau$ in S. Now

$$\lambda^j - \tau^j = \frac{2\mathrm{i}(z^j - x^j)}{(z^j + \mathrm{i})(x^j + \mathrm{i})} = \frac{2\mathrm{i}(z^j - x^j)}{(x^j + \mathrm{i})^2} + o(\|z - x\|)$$

as $z \to x$. Hence

$$h(z) = \xi + \sum_{j=1,2} \left(\frac{1 - \tau^j}{1 - \omega}\right)^2 \eta^j (z^j - x^j) + o(\|z - x\|)$$

as $z \to x$ in any nontangential approach region. That is, x is a C-point for h, and

$$\nabla h(x) = \frac{1}{(1-\omega)^2} \begin{pmatrix} (1-\tau^1)^2\eta^1 \\ (1-\tau^2)^2\eta^2 \end{pmatrix}. \tag{4.10}$$

In the case that τ is a facial B-point we have the following conclusion

Proposition 4.11. *Let $\tau \in \mathbb{T} \times \mathbb{D}$ be a B-point for $\varphi \in \mathcal{S}_2$ and suppose that $\tau^1 \neq 1$, $\varphi(\tau) \neq 1$. Then $x \in \mathbb{R} \times \Pi$ given by equations (4.3) is a C-point for $h \in \mathcal{P}_2$ given by equations (4.2), and*

$$\nabla h(x) = \left|\frac{1 - \tau^1}{1 - \omega}\right|^2 \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

where α is defined by equation (4.6).

For on combining equations (3.3) and (4.10) we find

$$\nabla h(x) = \frac{(1-\tau^1)^2}{(1-\omega)^2}\omega\overline{\tau^1}\alpha\begin{pmatrix}1\\0\end{pmatrix} = \left|\frac{1-\tau^1}{1-\omega}\right|^2\alpha\begin{pmatrix}1\\0\end{pmatrix}.$$

There is of course a converse to this result, which we shall not spell out.

We can derive a version of Theorem 3.2 for the Pick class.

Theorem 4.12. *If $x \in \mathbb{R}\times\Pi$ is a B-point for $h \in \mathcal{P}_2$ then every point of $\{x^1\}\times\Pi$ is a C-point for h, both h and ∇h are constant on $\{x^1\}\times\Pi$ and, for every $y \in \{x^1\}\times\Pi$,*

$$\nabla h(y) = \begin{pmatrix}\beta\\0\end{pmatrix} \tag{4.13}$$

where

$$\beta = \liminf_{z\to x}\frac{\operatorname{Im} h(z)}{\operatorname{Im} z^1}.$$

Proof. Apply Theorem 3.2 to the function $\varphi \in \mathcal{S}_2$ defined by equation (4.2). By Proposition 4.11 and equation (4.8) we have

$$\nabla h(y) = \nabla h(x) = \left|\frac{1-\tau^1}{1-\omega}\right|^2\alpha\begin{pmatrix}1\\0\end{pmatrix} = \left|\frac{1-\tau^1}{1-\omega}\right|^2\left|\frac{x^1+\mathrm{i}}{\xi+\mathrm{i}}\right|^2\beta\begin{pmatrix}1\\0\end{pmatrix}.$$

Since

$$(x^1+\mathrm{i})(1-\tau^1) = 2\mathrm{i} = (\xi+\mathrm{i})(1-\omega),$$

equation (4.13) follows. □

5 Julia reduction in $\mathcal{P}$

G. Julia, in the course of proving his celebrated Lemma in [8], introduced a form of reduction for functions in the Pick class and showed that reduction preserves the Pick class. His reduction process is analogous to the better-known Schur reduction for functions in the Schur class, but is associated with *boundary* points, that is, points on the real axis. Julia's reduction was subsequently used extensively by R. Nevanlinna, e.g. in [9]. In the next section we shall apply it to functions in $\mathcal{P}_2$ having a facial B-point.

Recall that, by the (one-variable) Julia-Carathéodory Theorem, if $x \in \mathbb{R}$ is a B-point for $f \in \mathcal{P}$ then f has an angular derivative $f'(x)$ at x. Furthermore, if f is non-constant,

$$f'(x) = \liminf_{z\to x}\frac{\operatorname{Im} f(z)}{\operatorname{Im} z} = \lim_{y\to 0+}\frac{\operatorname{Im} f(x+\mathrm{i}y)}{y} > 0.$$

Definition 5.1. (1) For any non-constant function $f \in \mathcal{P}$ and any $x \in \mathbb{R}$ such that x is a B-point for f we define the *reduction of f at x* to be the function g on Π given by the equation

$$g(z) = -\frac{1}{f(z)-f(x)} + \frac{1}{f'(x)(z-x)} \tag{5.2}$$

(2) For any $g \in \mathcal{P}$, any $x \in \mathbb{R}$ and any $a_0 \in \mathbb{R}, a_1 > 0$, we define the *augmentation of g at x by a_0, a_1* to be the function f on Π given by

$$\frac{1}{f(z) - a_0} = \frac{1}{a_1(z - x)} - g(z). \tag{5.3}$$

Note that in (1), since $f(x)$ is real and f is non-constant, the denominator $f(z) - f(x)$ is non-zero by the maximum principle. Note also that f defined by equation (5.3) is necessarily non-constant, for otherwise

$$\text{Im } g(z) = \text{const} + \frac{1}{a_1}\text{Im } \frac{1}{z - x},$$

and the last term can be an arbitrarily large negative number for $z \in \Pi$, contrary to the choice of $g \in \mathcal{P}$.

Reduction and augmentation are of course inverse operations.

The following is a refinement of Julia's result.

Theorem 5.4. *Let $x \in \mathbb{R}$.*

(1) *If x is a B-point for a non-constant function $f \in \mathcal{P}$ then the reduction of f at x also belongs to $\mathcal{P}$.*

(2) *If $g \in \mathcal{P}$ and $a_0 \in \mathbb{R}$, $a_1 > 0$ then the augmentation f of g at x by a_0, a_1 belongs to $\mathcal{P}$, has a B-point at x and satisfies $f(x) = a_0$, $f'(x) \leq a_1$. Moreover*

$$f'(x) = a_1 \Leftrightarrow \lim_{y \to 0+} yg(x + \mathrm{i}y) = 0.$$

Proof. Julia proved (1) in the case that f is regular at x (and observed in a footnote that it was true slightly more generally). One proof of Julia's result is given in [9], while a more elementary one, based on an idea of Schiffer and Bargmann, may be found in [4, Theorem 3.4].

For $\varepsilon > 0$ let $f_\varepsilon(z) = f(z + \mathrm{i}\varepsilon)$, $z \in \Pi$. Then $f_\varepsilon \in \mathcal{P}$ and f_ε is analytic at x. Hence, by Julia's result, the reduction g_ε of f_ε at x is in $\mathcal{P}$. Now

$$g_\varepsilon(z) = -\frac{1}{f(z + \mathrm{i}\varepsilon) - f(x + \mathrm{i}\varepsilon)} + \frac{1}{f'(x + \mathrm{i}\varepsilon)(z - x)}.$$

By the Julia-Carathéodory Theorem, $f(x + \mathrm{i}\varepsilon) \to f(x)$ and $f'(x + \mathrm{i}\varepsilon) \to f'(x)$ as $\varepsilon \to 0+$. Hence g_ε tends pointwise to g on Π as $\varepsilon \to 0+$. It follows that $g \in \mathcal{P}$.

To prove (2) consider any $g \in \mathcal{P}$, $a_0 \in \mathbb{R}$, $a_1 > 0$ and let f be the corresponding augmentation of g, so that equation (5.3) holds. For any $y > 0$,

$$\begin{aligned} \frac{\text{Im } f(x + \mathrm{i}y)}{y} &= \frac{1}{y}\text{Im } \left(a_0 + \frac{1}{\frac{1}{a_1 \mathrm{i}y} - g(x + \mathrm{i}y)} \right) \\ &= a_1 \text{Re } \frac{1}{1 - \mathrm{i}a_1 yg(x + \mathrm{i}y)}. \end{aligned} \tag{5.5}$$

For any $c \in (0, \infty)$ and $z \in \mathbb{C}$ we have

$$\begin{aligned} \text{Re } \frac{1}{z} \leq c \qquad &\Leftrightarrow \qquad \left| z - \frac{1}{2c} \right| \geq \frac{1}{2c} \\ &\Leftrightarrow \qquad z \notin D\left(\frac{1}{2c}, \frac{1}{2c}\right), \end{aligned} \tag{5.6}$$

where $D(w, r)$ denotes the open disk of centre w and radius r.

Now for any $y > 0$ we have

$$\text{Re } (1 - \mathrm{i}a_1 y g(x + \mathrm{i}y)) = 1 + a_1 y \text{Im } g(x + \mathrm{i}y) \geq 1,$$

and hence

$$1 - \mathrm{i}a_1 y g(x + \mathrm{i}y) \notin D(\tfrac{1}{2}, \tfrac{1}{2}).$$

It follows that

$$\text{Re } \frac{1}{1 - \mathrm{i}a_1 y g(x + \mathrm{i}y)} \leq 1$$

and hence, by equation (5.5), that

$$\frac{\text{Im } f(x + \mathrm{i}y)}{y} \leq a_1 \qquad \text{for all } y > 0.$$

Thus x is a B-point for f.

For $y > 0$ we have

$$\frac{1}{f(x + \mathrm{i}y) - a_0} = -\frac{1}{a_1}\left(\frac{\mathrm{i}}{y} + a_1 g(x + \mathrm{i}y)\right).$$

Now

$$\text{Im } \left(\frac{\mathrm{i}}{y} + a_1 g(x + \mathrm{i}y)\right) = \frac{1}{y} + a_1 \text{Im } g(x + \mathrm{i}y) \geq \frac{1}{y},$$

and hence

$$\frac{1}{|f(x + \mathrm{i}y) - a_0|} \to \infty \quad \text{ as } y \to 0+ .$$

Thus $f(x + \mathrm{i}y) \to a_0$ as $y \to 0+$, which is to say that $f(x) = a_0$.

Again by equation (5.3) we have, for $y > 0$,

$$\frac{\mathrm{i}y}{f(x + \mathrm{i}y) - a_0} = \frac{1}{a_1} - \mathrm{i}y g(x + \mathrm{i}y).$$

The left hand side tends to the limit $1/f'(x)$ as $y \to 0+$ and hence the limit of $\mathrm{i}y g(x + \mathrm{i}y)$ exists and is real. Now

$$\text{Re } \mathrm{i}y g(x + \mathrm{i}y) = -y \text{Im } g(x + \mathrm{i}y) \leq 0,$$

and hence

$$\frac{1}{f'(x)} = \frac{1}{a_1} + \lim_{y \to 0+} y \text{Im } g(x + \mathrm{i}y) \geq \frac{1}{a_1}. \tag{5.7}$$

It follows that $f'(x) \leq a_1$ and that $f'(x) = a_1$ if and only if

$$\lim_{y \to 0+} yg(x + \mathrm{i}y) = 0. \tag{5.8}$$

□

Examples of functions $g(z)$ in $\mathcal{P}$ for which the last displayed formula holds are $(z-x)^\alpha$ for $0 \leq \alpha \leq 1$, $-(z-x)^{-\alpha}$ for $0 < \alpha < 1$ and $\log(z-x)$. Functions in $\mathcal{P}$ for which it does *not* hold are $-1/(z-x)$ and $-\cot(z-x)$. Roughly speaking, condition (5.8) rules out poles of g at x.

6 Parametrization theorems

The first theorem describes the functions in $\mathcal{P}_2$ with a prescribed facial B-point.

Theorem 6.1. *Let $x \in \mathbb{R} \times \Pi$, $\xi \in \mathbb{R}$ and $\beta > 0$. The functions $h \in \mathcal{P}_2$ such that x is a B-point for h, $h(x) = \xi$ and $\nabla h(x) = (\beta, 0)^T$ are precisely the functions of the form*

$$h(z) = \xi + \frac{1}{\frac{1}{\beta(z^1 - x^1)} - g(z)} \tag{6.2}$$

for some function $g \in \mathcal{P}_2$ such that

$$\lim_{y \to 0+} yg(x^1 + \mathrm{i}y, x^2) = 0. \tag{6.3}$$

Proof. Consider a function h of the form (6.2) for some g as described. For fixed $z^2 \in \Pi$ it is clear that $g(., z^2)$ is in the one-variable Pick class $\mathcal{P}$ and $h(., z^2)$ is the augmentation of $g(., z^2)$ at x^1 by ξ, β. By Theorem 5.4(2), $h(., z^2) \in \mathcal{P}$, $h(., z^2)$ has a B-point at x^1 and

$$\lim_{y \to 0+} h(x^1 + \mathrm{i}y, z^2) = \xi.$$

Since $(x^1 + \mathrm{i}y, z^2)$ tends to the B-point (x^1, z^2) nontangentially as $y \to 0+$ we have $h(x^1, z^2) = \xi$. Again by Theorem 5.4, the angular derivative of $h(., z^2)$ at x^1 is at most β, and because of equation (6.3), when $z^2 = x^2$, this angular derivative is exactly β at x^1. It follows that $\nabla h(x) = (\beta, 0)^T$.

Conversely, suppose that x is a B-point for $h \in \mathcal{P}_2$, that $h(x) = \xi$ and that $\nabla h(x) = (\beta, 0)^T$. By Theorem 4.12, all points (x^1, w) with $w \in \Pi$ are C-points for h and we have $h(x^1, w) = \xi$, $\nabla h(x^1, w) = (\beta, 0)^T$. For any $w \in \Pi$ it follows that $h(., w)$ is a non-constant function in the Pick class, x^1 is a C-point for $h(., w)$ and the value and angular derivative of $h(., w)$ at x^1 are ξ and β respectively. We may therefore reduce $h(., w)$ at x^1 to obtain $g(., w) \in \mathcal{P}$ given by

$$g(z^1, w) = -\frac{1}{h(z^1, w) - \xi} + \frac{1}{\beta(z^1 - x^1)}.$$

Clearly g is analytic on Π^2, and so $g \in \mathcal{P}_2$. Now h is related to g by equation (6.2). Furthermore,

$$\frac{1}{\beta} = \frac{1}{D_1 h(x^1, w)} = \frac{1}{\beta} + \lim_{y \to 0+} yg(x^1 + \mathrm{i}y, w)$$

where D_1 here denotes the angular derivative in the first variable. Hence

$$\lim_{y\to 0+} yg(x^1 + \mathrm{i}y, w) = 0$$

for all $w \in \Pi$, and in particular when $w = x^2$. Thus g satisfies equation (6.3). $\square$

Remark 6.4. The proof shows that for any $g \in \mathcal{P}_2$, if $x^1 \in \mathbb{R}$ and $\lim_{y\to 0+} yg(x^1 + \mathrm{i}y, z^2) = 0$ holds for some $z^2 \in \Pi$ then it holds for all $z^2 \in \Pi$.

Remark 6.5. The parametrization formula (6.2), where g is allowed to be a *free* function in the Pick class $\mathcal{P}_2$, parametrizes the solutions of a relaxed one-point interpolation problem. That is, in Theorem 6.1 one may omit the condition (6.3) on g and replace the condition $\nabla h(x) = (\beta, 0)$ by: $\nabla h(x) = (\beta', 0)$ for some $\beta' \leq \beta$.

We can now invoke Cayley transformation to obtain a parametrization of functions in $\mathcal{S}_2$ with prescribed value and gradient at a facial B-point.

Theorem 6.6. *Let $\tau \in \mathbb{T} \times \mathbb{D}$, $\omega \in \mathbb{T}$ and $\alpha > 0$. Suppose that $\tau^1 \neq 1$, $\omega \neq 1$. The functions $\varphi \in \mathcal{S}_2$ such that τ is a B-point for φ, that $\varphi(\tau) = \omega$ and that $\nabla\varphi(\tau) = (\omega\overline{\tau^1}\alpha, 0)^T$ are precisely the functions of the form*

$$\varphi(\lambda) = \frac{h(z) - \mathrm{i}}{h(z) + \mathrm{i}} \tag{6.7}$$

where

$$z^j = \mathrm{i}\frac{1+\lambda^j}{1-\lambda^j} \quad \text{for } j = 1, 2,$$

and

$$h(z) = \mathrm{i}\frac{1+\omega}{1-\omega} + \frac{1}{\frac{1}{\beta(z^1 - x^1)} - g(z)} \tag{6.8}$$

where

$$\beta = \frac{1 - \operatorname{Re}\tau^1}{1 - \operatorname{Re}\omega}\,\alpha, \qquad x^1 = \mathrm{i}\frac{1+\tau^1}{1-\tau^1}$$

and g is a function in $\mathcal{P}_2$ such that

$$\lim_{y\to 0+} yg\left(\mathrm{i}\frac{1+\tau^1}{1-\tau^1} + \mathrm{i}y, \mathrm{i}\frac{1+\tau^2}{1-\tau^2}\right) = 0. \tag{6.9}$$

Proof. Consider a function φ of the form (6.7) with h as described. By Theorem 6.2 we have $h \in \mathcal{P}_2$,

$$x = (x^1, x^2) = \left(\mathrm{i}\frac{1+\tau^1}{1-\tau^1}, \mathrm{i}\frac{1+\tau^2}{1-\tau^2}\right)$$

is a B-point for h,

$$h(x) = \mathrm{i}\frac{1+\omega}{1-\omega} \qquad \text{and} \qquad \nabla h(x) = \begin{pmatrix}\beta \\ 0\end{pmatrix}.$$

It follows that τ is a B-point for φ, $\varphi(\tau) = \omega$ and, by equation (4.10)

$$\begin{aligned}\nabla\varphi(\tau) = \begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix} &= \left(\frac{1-\omega}{1-\tau^1}\right)^2 \nabla h(x) = \left(\frac{1-\omega}{1-\tau^1}\right)^2 \begin{pmatrix}\beta\\ 0\end{pmatrix}\\ &= \left(\frac{1-\omega}{1-\tau^1}\right)^2 \frac{1-\operatorname{Re}\,\tau^1}{1-\operatorname{Re}\,\omega} \begin{pmatrix}\alpha\\ 0\end{pmatrix}\\ &= \omega\overline{\tau^1} \begin{pmatrix}\alpha\\ 0\end{pmatrix}\end{aligned}$$

as required.

Conversely, if τ is a B-point for $\varphi \in \mathcal{S}_2$, if $\varphi(\tau) = \omega$ and $\nabla\varphi(\tau) = (\omega\overline{\tau^1}\alpha, 0)^T$ then h defined by equation (6.7) belongs to $\mathcal{P}_2$, x is a B-point for h and $h(x) = \mathrm{i}(1+\omega)/(1-\omega)$, $\nabla h(x) = (\beta, 0)^T$. We may therefore apply Theorem 6.2 to obtain the parametric expression (6.8) for h. □

Remark 6.10. Again there is a "relaxed" version of the parametrization. In Theorem 6.6, if one enlarges the class of φ by replacing the condition $\nabla\varphi(\tau) = (\omega\overline{\tau^1}\alpha, 0)^T$ by $\nabla\varphi(\tau) = (\omega\overline{\tau^1}\alpha', 0)^T$ for some α' with $0 < \alpha' \leq \alpha$, then one obtains the same parametrization but without the limit condition (6.9) on g.

7 An interpolation problem

Theorem 3.2 suggests a natural interpolation problem: to describe the functions $\varphi \in \mathcal{S}_2$ having B-points at finitely many given points in $\mathbb{T} \times \mathbb{D}$ and $\mathbb{D} \times \mathbb{T}$ and with prescribed values of φ and $\nabla\varphi$ at those points. In view of Corollary 3.13, if there are interpolation nodes on both types of face then the target values of φ must all be the same, but the target values of $\nabla\varphi$ may differ. The parametrization theorems allow us to solve this problem. We state the result for the half-plane version. Taking a slight liberty with terminology of D. Sarason, we define

Problem ∂NP$\mathcal{P}_2$ (facial): *Given $x_1, \ldots, x_m \in \mathbb{R} \times \Pi$, $y_1, \ldots, y_n \in \Pi \times \mathbb{R}$ for some $m, n \geq 0$ and $\xi \in \mathbb{R}$, $\beta_1, \ldots, \beta_m > 0, \gamma_1, \ldots, \gamma_n > 0$, determine whether there exists a function h in the two-variable Pick class $\mathcal{P}_2$ that satisfies, for $j = 1, \ldots, m$, $k = 1, \ldots, n$,*

(1) *x_j and y_k are B-points for h*

(2) *$h(x_j) = \xi = h(y_k)$ and*

(3) $$\nabla h(x_j) = \begin{pmatrix}\beta_j\\ 0\end{pmatrix}, \qquad \nabla h(y) = \begin{pmatrix}0\\ \gamma_k\end{pmatrix}.$$

Describe the set of all such functions h when they exist.

We shall write $e_1 = (1, 0)$, $e_2 = (0, 1)$ for the standard basis in $\mathbb{C}^2$.

Theorem 7.1. *Problem $\partial NP\mathcal{P}_2$ (facial) always has infinitely many solutions. The solution set consists of all functions h expressible in the form*

$$h(z) = \xi + \frac{1}{r(z) - f(z)} \qquad \text{for all } z \in \Pi \tag{7.2}$$

for some $f \in \mathcal{P}_2$ such that

$$\lim_{t\to 0+} tf(x^j + \mathrm{i}te_1) = 0 = \lim_{t\to 0+} tf(y^k + \mathrm{i}te_2), \quad 1 \le j \le m,\ 1 \le k \le n, \tag{7.3}$$

where

$$r(z) = \sum_{j=1}^{m} \frac{1}{\beta_j(z^1 - x_j^1)} + \sum_{k=1}^{n} \frac{1}{\gamma_k(z^2 - y_k^2)}.$$

In particular, the function $h = \xi + 1/(r-c)$ is a solution of Problem $\partial NP\mathcal{P}_2$ (facial) for any $c \in \mathbb{R}$.

Proof. We prove necessity by induction on $m + n$. The assertion holds trivially if $m = n = 0$, where empty sums are as usual defined to be 0. Suppose $m > 0$ and that necessity holds for $m + n - 1$. Let $h \in \mathcal{P}_2$ satisfy conditions (1)-(3). By Theorem 6.1 there exists $g \in \mathcal{P}_2$ such that

$$\lim_{t\to 0+} tg(x^1 + \mathrm{i}te_1) = 0 \tag{7.4}$$

and

$$h(z) = \xi + \frac{1}{\frac{1}{\beta_1(z^1 - x_1^1)} - g(z)} \qquad \text{for all } z \in \Pi. \tag{7.5}$$

Let $\tilde{g} = -1/g$; then $g \in \mathcal{P}_2$, and we claim that $x_2, \ldots, x_m, y_1, \ldots, y_n$ are B-points for $\tilde{g}$. Indeed, by Theorem 3.2, all these points are C-points for h, and in view of the hypotheses (2) and (3) on h,

$$h(x_j + \mathrm{i}te_1) = \xi + \beta_j \mathrm{i}t + o(t), \quad h(y_k + \mathrm{i}te_2) = \xi + \gamma_k \mathrm{i}t + o(t) \quad \text{as } t \to 0+.$$

Hence, for $j = 2, \ldots, m$,

$$g(x_j + \mathrm{i}te_1) = -\frac{1}{\beta_j t + o(t)} + \frac{1}{\beta_1(x_j^1 + \mathrm{i}t - x_1^1)},$$

from which we easily calculate

$$\frac{\tilde{g}(x_j + \mathrm{i}te_1)}{t} = \beta_j + o(1).$$

Thus $x_2, \ldots, x_m$ are B-points for $\tilde{g}$, and $\tilde{g}(x_j) = 0$, $j = 2 \ldots, m$. Likewise $y_1, \ldots, y_m$ are B-points for $\tilde{g}$, and $\tilde{g}$ takes the value 0 at the y_k. Moreover, by Theorem 3.2,

$$\nabla\tilde{g}(x_j) = \begin{pmatrix} \beta_j \\ 0 \end{pmatrix}, \quad \nabla\tilde{g}(y_k) = \begin{pmatrix} 0 \\ \gamma_k \end{pmatrix}$$

for $j = 2, \ldots, m,\ k = 1, \ldots, n$.

By the inductive hypothesis there exists $f \in \mathcal{P}_2$ such that

$$\lim_{t \to 0+} f(x_j + \mathrm{i}te_1) = 0 = \lim_{t \to 0+} f(y_k + \mathrm{i}te_2)$$

for $j = 2, \ldots, m,\ k = 1, \ldots, n$ and

$$\tilde{g}(z) = 0 + \frac{1}{r_1(z) - f(z)} \qquad \text{for all } z \in \Pi$$

where

$$r_1(z) = \sum_{j=2}^{m} \frac{1}{\beta_j (z^1 - x_j^1)} + \sum_{k=1}^{n} \frac{1}{\gamma_k (z^2 - y_k^2)}.$$

Thus

$$g(z) = -\frac{1}{\tilde{g}(z)} = f(z) - r_1(z).$$

This relation in conjunction with equation (7.4) tells us that

$$\lim_{t \to 0+} tf(x_1 + \mathrm{i}te_1) = 0,$$

so that f satisfies conditions (7.3). On substituting for g in equation (7.5) we obtain

$$\begin{aligned} h(z) &= \xi + \frac{1}{\frac{1}{\beta_1 (z^1 - x_1^1)} + r_1(z) - f(z)} \\ &= \xi + \frac{1}{r(z) - f(z)}, \end{aligned}$$

which is the desired relation (7.2). We have proved necessity in Theorem 7.1.

Conversely, suppose that h is expressible in the form (7.2), where $f \in \mathcal{P}_2$ satisfies conditions (7.3). Observe that, for $t > 0$,

$$\begin{aligned} tr(x_j + \mathrm{i}te_1) &= \sum_{\ell=1}^{m} \frac{t}{\beta_\ell (x_j^1 + \mathrm{i}t - x_\ell^1)} + \sum_{k=1}^{n} \frac{t}{\gamma_k (x_j^2 - y_k^2)} \\ &= \frac{1}{\beta_j \mathrm{i}} + o(1). \end{aligned}$$

Hence

$$\begin{aligned} \frac{\mathrm{Im}\ h(x_j + \mathrm{i}te_1)}{t} &= \frac{1}{t} \mathrm{Im}\ \frac{1}{r(x_j + \mathrm{i}te_1) - f(x_j + \mathrm{i}te_1)} \\ &= \mathrm{Im}\ \frac{1}{\frac{1}{\beta_j \mathrm{i}} + o(1) - tf(x_j + \mathrm{i}te_1)}. \end{aligned}$$

In view of the relation (7.3), it follows that

$$\lim_{t \to 0+} \frac{\mathrm{Im}\ h(x_j + \mathrm{i}te_1)}{t} = \beta_j, \quad j = 1, \ldots, m,$$

so that x_j is a B-point for h and $\nabla h(x_j) = (\beta_j, 0)^T$. Similarly, y_k is a B-point for h and $\nabla h(y_k) = (0, \gamma_k)$ for $k = 1, \ldots, n$. Hence h satisfies (1)-(3). In particular we can take f to be a real constant c, yielding the solution $h = \xi + 1/(r - c)$. □

Remark 7.6. We can also consider a relaxed version of the interpolation problem: **Problem ∂NP$\mathcal{P}_2'$ (facial)**: *As Problem $\partial NP\mathcal{P}_2$ (facial), but with condition* (3) *replaced by*

$$(3') \qquad \nabla h(x_j) = \begin{pmatrix} \beta_j' \\ 0 \end{pmatrix}, \qquad \nabla h(y) = \begin{pmatrix} 0 \\ \gamma_k' \end{pmatrix}$$

for some β_j', γ_k' such that $0 < \beta_j' \leq \beta_j$, $0 < \gamma_k' \leq \gamma_k$.

The general solution of this relaxed problem is again given by the formula (7.2), but now without the limit condition (7.3) on f.

Do analogous results hold for the polydisk in dimensions greater than 2? They may well do, but our methods, depending as they do on the use of models, only give analogous statements for those functions which possess models in the sense of Definition 2.1 (modified appropriately). In dimensions higher than 2 such functions constitute a class that is strictly smaller than the Schur class [2].

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