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Loewner's theorem on monotone matrix functions, by Barry Simon, Grundlehren der mathematischen Wissenschaften, Vol. 354, Springer Nature Switzerland, 2019, xi+459 pp., ISBN 978-3-030-22421-9, 978-3-030-22422-6 (ebook)

1. History

If A is a self-adjoint matrix, or more generally a self-adjoint operator on a Hilbert space of any dimension, we say it is *positive*, and write $A \geq 0$, if $\langle Av, v\rangle \geq 0$ for all vectors v; this is equivalent to the spectrum $\sigma(A)$ lying in $[0, \infty)$. This notion allows us to put a partial order on all self-adjoint operators acting on a given space, namely

$$A \leq B \;\Leftrightarrow\; B - A \geq 0.$$

Loewner asked the question, "What functions preserve this order?" To make the question precise, we must first decide what we mean by a function applied to a self-adjoint operator. This is called functional calculus, and there are several ways to do it; the good news is that all reasonable methods coincide.

To start, let us assume that the underlying Hilbert space is finite dimensional, so A is a self-adjoint matrix, and let f be any real-valued function defined on the spectrum of A. The easiest way to define $f(A)$ is to choose a basis of eigenvectors of A which diagonalizes the matrix and then apply f to each diagonal entry separately. Alternatively, one can choose a polynomial p that agrees with f on $\sigma(A)$ and define $f(A) = p(A)$. If now we let A be a bounded self-adjoint operator on an infinite dimensional Hilbert space, we may no longer be able to diagonalize it exactly; but the spectral theorem says we can essentially do this using measure theory (see, e.g., [4, 13, 19] for an exact statement and proof), and define $f(A)$ for any bounded measurable function. In particular, if f is a continuous function on $\sigma(A)$, by Weierstrass's theorem we can choose a sequence of polynomials p_k that converges uniformly to f on $\sigma(A)$, and we can define $f(A)$ to be the norm limit of the sequence $p_k(A)$.

Let $\mathbb{M}_n$ denote the space of n-by-n complex matrices, and $\mathrm{SA}\mathbb{M}_n$ the self-adjoint ones.

Definition. For each positive integer n and each nonempty interval $(a, b) \subseteq \mathbb{R}$, let $\mathcal{M}_n(a, b)$ denote the set of continuous functions $f : (a, b) \to \mathbb{R}$ with the property that if A and B are both in $\mathrm{SA}\mathbb{M}_n$ with $\sigma(A) \cup \sigma(B) \subset (a, b)$, then

$$A \leq B \;\Rightarrow\; f(A) \leq f(B).$$

The functions in $\mathcal{M}_n(a, b)$ are called n-matrix monotone. When $n = 1$ they are just the increasing functions on the interval, a class studied extensively in one's first rigorous calculus class. In [15] Loewner characterized the n-matrix monotone functions for $n \geq 2$, showed that for each n

$$\mathcal{M}_{n+1}(a, b) \subsetneq \mathcal{M}_n(a, b),$$

and furthermore characterized $\bigcap_{n=1}^{\infty} \mathcal{M}_n(a, b)$, the operator monotone functions. (An approximation argument shows that $\bigcap_{n=1}^{\infty} \mathcal{M}_n(a, b)$ is precisely the class of

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functions that preserve order on the set of all self-adjoint operators with spectrum in (a, b).)

Definition. For each positive integer n and each nonempty interval $(a, b) \subseteq \mathbb{R}$, let $\mathcal{L}_n(a, b)$ denote the set of differentiable functions $f : (a, b) \to \mathbb{R}$ with the property that, for any n distinct points $x_1, \ldots, x_n \in (a, b)$, the divided difference matrix

$$L_{ij} = \begin{cases} \frac{f(x_i)-f(x_j)}{x_i-x_j} & i \neq j \\ \\ f'(x_i) & i = j \end{cases}$$

is positive.

Theorem 1.1 (Loewner's theorem)**.**

(i) *For each $n \geq 2$, the classes $\mathcal{M}_n(a, b)$ and $\mathcal{L}_n(a, b)$ coincide.*
(ii) *Moreover, a function $f : (a, b) \to \mathbb{R}$ is in $\bigcap_{n=1}^{\infty} \mathcal{M}_n(a, b)$ if and only if there is a holomorphic function F that maps the upper half-plane to itself and such that $\lim_{y\to 0} F(x + iy) = f(x)$ for every $x \in (a, b)$.*

Using the theorem, one can see that the function $x^{1/3}$ is n-monotone for all n, but x^3 is not 2-monotone on any interval containing 0. There are several intriguing aspects of Loewner's theorem. Part (i) is basically real analysis—the condition of being in $\mathcal{L}_n(a, b)$ is related to concavity. When $n = 2$, this is made precise by the following theorem of Donoghue [7]. (Donoghue attributes the theorem to Dobsch [6], but Simon points out in his notes to Section 14 that he can find no mention of 2-by-2 matrices in Dobsch's paper.)

Theorem 1.2 (Donoghue and Dobsch)**.** *A nonconstant real-valued function f is in $\mathcal{M}_2(a, b)$ if and only if*

(i) *f is C^1 and $f'(x) > 0$ for all $x \in (a, b)$.*
(ii) *$(f')^{-1/2}$ is concave.*

Part (ii) of Loewner's theorem, however, passes into complex analysis and gives a testable condition for a function to have a holomorphic extension that maps the upper half-plane to itself, namely all the divided difference matrices must be positive. It ties into earlier work of Nevanlinna, who gave a complete description of all such functions [16]; we shall use $\mathbb{C}_+$ to denote the upper half-plane.

Theorem 1.3 (Nevanlinna)**.** *Let f be a holomorphic function defined on $\mathbb{C}_+$. Then the range of f is contained in $\overline{\mathbb{C}_+}$ if and only if there is a constant $A \geq 0$ and a finite positive measure μ on $\mathbb{R}$ so that*

$$f(z) = \Re f(i) + Az + \int \frac{1 + tz}{t - z} d\mu(t). \tag{1.4}$$

Moreover, f is an analytic extension of a real-valued function defined on the interval $(a, b) \subseteq \mathbb{R}$ if and only if μ puts no mass on (a, b).

2. Related results and generalizations

We shall say that a real-valued function f on (a, b) is n-matrix convex if whenever A and B are both in $\mathrm{SA}\mathbb{M}_n$ with $\sigma(A) \cup \sigma(B) \subset (a, b)$, then

$$f(tA + (1 - t)B) \leq tf(A) + (1 - t)f(B) \qquad \forall\, t \in [0, 1].$$

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The set of n-matrix convex functions is closely related to the n-monotone ones; they were first studied by Loewner's student Kraus [12]. Define

$$f_y^{(1)}(x) \;=\; \begin{cases} \frac{f(x)-f(y)}{x-y} & x \neq y \\ \\ f'(x) & x = y. \end{cases}$$

Theorem 2.1 (Kraus)**.** *Let f be C^2 on (a,b). Then f is n-matrix convex on (a,b) if and only if $f_y^{(1)} \in \mathcal{M}_n(a,b)$ for all $y \in (a,b)$.*

Recently, Heinävaara proved [9] that n-matrix convexity is, like n-matrix monotonicity, a local property—if a function f has the property on (a,b) and (c,d), and $a < c < b < d$, then f has the property on (a,d).

Part (ii) of Loewner's theorem describes when a function defined on an arc of the boundary of $\mathbb{C}_+$ extends to a holomorphic map from $\mathbb{C}_+$ to itself. Fitzgerald and Loewner asked an even more audacious question [8, 14], "When is this extension univalent?"

Theorem 2.2 (Fitzgerald and Loewner)**.** *A function $f : (a,b) \to \mathbb{R}$ extends to a univalent function from $\mathbb{C}_+$ to $\mathbb{C}_+$ if and only if it satisfies $f'(x) > 0$ for every x and, for each divided difference matrix, the matrix obtained by taking the entry-wise logarithm is positive.*

Let $\mathrm{SA}\mathbb{M}_n^d$ denote the set of all d-tuples of matrices in $\mathrm{SA}\mathbb{M}_n$, and let $\mathrm{CSA}\mathbb{M}_n^d$ denote the set of d-tuples that pairwise commute. If $A = (A^1, \ldots, A^d)$ and $B = (B^1, \ldots, B^d)$ are in $\mathrm{SA}\mathbb{M}_n^d$, we say $A \leq B$ if $A^r \leq B^r$ for each $1 \leq r \leq d$. If $A \in \mathrm{CSA}\mathbb{M}_n^d$, the spectral theorem allows a functional calculus as before; we can choose a basis of joint eigenvectors that simultaneously diagonalizes all of them, so that A^r is the diagonal matrix with entries $\lambda_1^r, \ldots, \lambda_n^r$. The spectrum of A, again denoted $\sigma(A)$, is the n-tuple of points $\lambda_j = (\lambda_j^1, \ldots, \lambda_j^d)$ in $\mathbb{R}^d$. Then we can define $f(A)$ to be the diagonal matrix with entries $f(\lambda_j)$. We could instead use the polynomial trick from earlier, approximating f by polynomials; this method also works to apply a continuous function of d variables to a commuting d-tuple of self-adjoint operators.

Definition. Let R be an open box in $\mathbb{R}^d$, and let $f : R \to \mathbb{R}$ be a differentiable function. We say f is locally n-monotone on R if, whenever $S(t)$ is a differentiable path in $\mathrm{CSA}\mathbb{M}_n^d$ with $\sigma(S(0)) \subseteq R$ and $S'(0) \geq 0$, then

$$\frac{d}{dt} f(S(t))\Big|_{t=0} \;\geq\; 0.$$

We say f is globally n-monotone on R if, whenever A and B are in $\mathrm{CSA}\mathbb{M}_n^d$ with $\sigma(A) \cup \sigma(B) \subseteq R$, then

$$A \leq B \;\Rightarrow\; f(A) \leq f(B).$$

When $d = 1$, locally and globally monotone are the same, because if $A \leq B$, then $S(t) = (1-t)A + tB$ is an increasing path from A to B, and by the fundamental theorem of calculus,

$$f(B) - f(A) \;=\; \int_0^1 \frac{d}{dt} f(S(t))dt.$$

In several variables, however, the d-tuple $((1-t)A_1 + tB_1, \ldots, (1-t)A_d + tB_d)$ is not in general commutative, so it is not clear how to make sense of $f(S(t))$.

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Loewner's theorem generalizes completely to the locally monotone case [2]. The requirement that F maps $\mathbb{C}_+$ to $\mathbb{C}_+$ becomes that F maps every d-tuple of commuting operators with positive imaginary parts to an operator with positive imaginary part. For $d = 1$ or 2, this is the same as requiring that $F : \mathbb{C}_+^d \to \overline{\mathbb{C}_+}$, by the Cayley transform of the von Neumann inequality [22] and Andô inequality [3], respectively; but it is a strictly stronger requirement when $d \geq 3$ [5, 21].

Theorem 2.3. *Let R be an open box in $\mathbb{R}^d$, and let $f : R \to \mathbb{R}$ be differentiable.*

(i) *For each $n \geq 2$, f is locally n-monotone on R if and only if for any n distinct points $x_1, \ldots, x_n$ in R, there are d positive n-by-n matrices $\Gamma^r, 1 \leq r \leq d$, with $\Gamma^r_{ii} = \frac{\partial f}{\partial x^r}|_{x_i}$ so that*

$$f(x_j) - f(x_i) \;=\; \sum_{r=1}^{d} (x_j^r - x_i^r)\Gamma^r_{ij}. \tag{2.4}$$

(ii) *Moreover, f is locally n-monotone for every n if and only if there is a holomorphic function F that maps $\mathbb{C}_+^d$ to $\overline{\mathbb{C}_+}$, such that $\lim_{y\to 0} F(x+iy) = f(x)$ for every $x \in R$, and such that F maps d-tuples of commuting operators with positive imaginary parts to operators with positive imaginary part.*

Observe that when $d = 1$, condition (2.4) is the same as requiring the divided difference matrix to be positive. Whether locally and globally monotone are the same in several variables is an open question: it is shown in [2] that a rational function of two variables that is locally n-monotone for all n is also globally monotone, but whether this is true more generally is unknown. Simon calls the forward direction of part (ii) of Loewner's theorem the *hard direction* and calls the reverse implication the *easy direction*. In this parlance, for globally monotone functions in $d > 1$ variables, the hard direction is true, and the easy one is unknown.

Another possible way to generalize Loewner's theorem to several variables is to use noncommuting functions. A noncommuting function of d variables is a function that sends d-tuples of n-by-n matrices to an n-by-n matrix and preserves direct sums and similiarities. The domain of such a function is a subset of $\bigcup_n \mathbb{M}_n^d$ that is closed with respect to direct sums. J. L. Taylor realized that these were the key properties to abstract from noncommutative polynomials to get something akin to a noncommutative holomorphic function [20]. Interest in noncommutative functions has grown dramatically; see the monograph [11] for a detailed account or [1] for a brief survey.

A noncommutative analogue of Loewner's theorem has been proved by Pascoe and Tully-Doyle [17]. Let

$$\Pi^d \;=\; \bigcup_n \{Z \in \mathbb{M}_n^d : \Im(Z^r) > 0,\ 1 \leq r \leq d\}.$$

Theorem 2.5 (Pascoe and Tully-Doyle)**.** *Let $R = (a^1, b^1) \times \cdots \times (a^d, b^d)$ be an open box in $\mathbb{R}^d$, and let*

$$D = \bigcup_n \{A \in \mathrm{SAM}\mathbb{M}_n^d : \sigma(A^r) \subset (a^r, b^r),\ 1 \leq r \leq d\}.$$

A noncommutative function f that maps D to self-adjoint matrices is matrix monotone for all n if and only if f analytically continues to a noncommutative function from Π^d to Π^1.

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3. The book

Simon's book studies Loewner's theorem in one variable, and gives eleven different proofs of the hard direction. As he puts it, the book is a love poem to Loewner's theorem.

Part I of the book is called *Tools*, and here he introduces Nevanlinna's theorem, which he calls the Herglotz representation theorem (since it follows from a 1911 theorem of Herglotz for functions on the disk [10]), and he uses it to prove the easy direction. He then proves part (i) of Loewner's theorem and follows with an extensive treatment of matrix convex functions.

Part II starts by giving 4 different proofs of Pick's theorem (and later a fifth), a theorem first proved by Pick in 1916 [18]. Pick's theorem describes when a function mapping the upper half-plane to itself and satisfying given interpolation conditions exists; part (ii) of Loewner's theorem is a boundary value version.

Theorem 3.1 (Pick)**.** *Let $z_1, \dots, z_n$ and $w_1, \dots, w_n$ be points in $\mathbb{C}_+$. A necessary and sufficient condition for the existence of a holomorphic function $F : \mathbb{C}_+ \to \mathbb{C}_+$ satisfying $F(z_j) = w_j, 1 \le j \le n$, is that the Pick matrix*

$$P_{ij} = \frac{w_i + \overline{w_j}}{z_i + \overline{z_j}}$$

be positive.

The author's first proof of Loewner's theorem is as a limiting case of Pick's theorem. He goes on to give Loewner's original proof, a proof using moment theory by Bendat and Sherman, a Hilbert space proof due to Koranyi, and Krein–Milman based proofs by Hansen–Pedersen and Hansen, Sparr's proof using positive functions, Ameur's proof using quadratic interpolation, the Wigner–von Neumann proof using continued fractions, a new proof by the author also using continued fractions, a Hardy space proof by Rosenblum and Rovnyak, and a Mellin transform proof by Boutet de Monville.

In Part III, applications to three areas are discussed: operator means, quantum strong subadditivity, and unitarily invariant norm inequalities.

The book is very carefully written. For each proof, the author gives all the necessary background material and includes detailed historical remarks. It is an impressive *tour-de-force*.

I would consider this book a mathematics lounge book—the sort of book that should be left lying around the lounge, so that one can dip into it over coffee, read a section, and then go back to work. It is accessible and is a delight to read.

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John E. McCarthy
Department of Mathematics and Statistics
Washington University in St. Louis
St. Louis, Missouri
Email address: mccarthy@wustl.edu

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