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Contemporary Mathematics

Shining a Hilbertian lamp on the bidisk

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Abstract. Four lectures on how different aspects of the function theory of the bidisk can be illuminated by using Hilbert spaces and operator theory.

1. Lecture 1: Model Theory

The basic idea behind model theory is to associate a Hilbert space construction with a function, and then use Hilbert space theory to illuminate the function theory.

In one variable, one approach is to study the de Branges-Rovnyak space associated with a function ϕ in the ball of $H^\infty(\mathbb{D})$. This is the Hilbert space of analytic functions on the disk $\mathbb{D}$ with reproducing kernel

$$\frac{1-\overline{\phi(\lambda)}\phi(\zeta)}{1-\bar{\lambda}\zeta}. \tag{1.1}$$

A nice exposition is in the book [**20**] by D. Sarason.

Definition 1.2. We will say that k is a kernel on X, or equivalently that k is positive semi-definite on X, written $k \geq 0$, if k is a function from $X \times X$ to $\mathbb{C}$ such that, for any finite set of distinct points $x_1, \ldots, x_N$ in X, the matrix $[k(x_i, x_j)]$ is positive semi-definite, which means that for any complex numbers $c_1, \ldots, c_N$ we have

$$\sum_{i,j=1}^{N} c_i \bar{c}_j k(x_i, x_j) \ \geq \ 0.$$

Notice that saying that (1.1) is a kernel on $\mathbb{D}$ is equivalent to saying that ϕ is in the (closed) unit ball of $H^\infty(\mathbb{D})$. Indeed, let H^2 be the Hardy space, and

$$k^S(\zeta, \lambda) \ = \ k^S_\lambda(\zeta) \ = \ \frac{1}{1-\bar{\lambda}\zeta} \tag{1.3}$$

be the Szegő kernel on H^2. Let M_ϕ be the operator of multiplication by ϕ. It is straightforward to check that

$$M^*_\phi k^S_\lambda \ = \ \overline{\phi(\lambda)} k^S_\lambda.$$

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We have

$$\begin{aligned}
\|\phi\| \le 1 &\Leftrightarrow \|M_\phi\| \le 1 \\
&\Leftrightarrow I - M_\phi M_\phi^* \ \ge\ 0 \\
&\Leftrightarrow \langle (I - M_\phi M_\phi^*) \sum_i c_i k^S_{\lambda_i}, \sum_j c_j k^S_{\lambda_j} \rangle \ \ge\ 0 \quad \forall\ c_i, \lambda_i \\
&\Leftrightarrow \sum_{i,j} c_i \bar{c}_j \left(1 - \overline{\phi(\lambda_i)}\phi(\lambda_j)\right) \langle k^S_{\lambda_i}, k^S_{\lambda_j} \rangle \ \ge\ 0 \quad \forall\ c_i, \lambda_i \\
&\Leftrightarrow \left[\frac{1 - \overline{\phi(\lambda_i)}\phi(\lambda_j)}{1 - \bar{\lambda}_i \lambda_j}\right] \ \ge\ 0 \qquad \forall\ \lambda_i. \qquad \lhd
\end{aligned}$$

Given a kernel k on X, it is an important fact that one can always realize it as a Grammian, *i.e.* one can find a Hilbert space $\mathcal{H}$ and a map $u : X \to \mathcal{H}$ so that

$$k(x,y) \ =\ \langle u(x), u(y) \rangle \ :=\ \langle u_x, u_y \rangle.$$

So if (1.1) is positive semidefinite, we can write

$$\frac{1 - \overline{\phi(\lambda)}\phi(\zeta)}{1 - \bar{\lambda}\zeta} \ =\ \langle u_\zeta, u_\lambda \rangle_{\mathcal{H}}. \tag{1.4}$$

Now inside (1.4) lurks an isometry. Indeed, define $V : \mathbb{C} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathcal{H}$ by

$$V \ :\ \begin{pmatrix} 1 \\ \zeta u_\zeta \end{pmatrix} \ \mapsto\ \begin{pmatrix} \phi(\zeta) \\ u_\zeta \end{pmatrix}.$$

Then equation (1.4) is equivalent to the assertion that V is an isometry on the linear span of vectors of the form

$$\begin{pmatrix} 1 \\ \zeta_i u_{\zeta_i} \end{pmatrix} \qquad \zeta_i \ \in\ \mathbb{D}.$$

If the codimension of the range is at least as large as the codimension of the domain, then V can be extended to an isometry on all of $\mathbb{C} \oplus \mathcal{H}$. If the codimension is smaller, the same effect can be achieved by adding an infinite dimensional summand to $\mathcal{H}$. Thus we have essentially proved the following realization formula; see *e.g.* [**9**] or [**5**] for full details.

THEOREM 1.5. *The function ϕ is in the closed unit ball of $H^\infty(\mathbb{D})$ if and only if there is a Hilbert space $\mathcal{H}$ and an isometry $V : \mathbb{C} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathcal{H}$, such that, writing V as*

$$V \ =\ \begin{array}{c} \\ \mathbb{C} \\ \mathcal{H} \end{array} \begin{array}{c} \begin{array}{cc} \mathbb{C} & \mathcal{H} \end{array} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}, \tag{1.6}$$

one has

$$\phi(\lambda) \ =\ A + \lambda B (I - \lambda D)^{-1} C. \tag{1.7}$$

This theory was generalized to the bidisk by Jim Agler [**3**]. We shall use superscripts to denote coordinates; so a point λ in $\mathbb{D}^2$ will be written $\lambda = (\lambda^1, \lambda^2)$. In lieu of studying the positive semi-definite form (1.1), Agler proved:

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THEOREM 1.8. *Let $\phi : \mathbb{D}^2 \to \mathbb{D}$ be a function. Then ϕ is analytic iff there are kernels Γ and Δ on $\mathbb{D}^2$ so that*

$$1 - \overline{\phi(\mu)}\phi(\lambda) \;=\; (1 - \bar{\mu}^1\lambda^1)\Gamma(\lambda,\mu) \,+\, (1 - \bar{\mu}^2\lambda^2)\Delta(\lambda,\mu). \tag{1.9}$$

The realization formula becomes:

THEOREM 1.10. *The function ϕ is in the closed unit ball of $H^\infty(\mathbb{D}^2)$ if and only if there are auxiliary Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and an isometry*

$$V \,:\, \mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \;\to\; \mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$$

such that, if $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$, V is written as

$$V \;=\; \begin{matrix} \\ \mathbb{C} \\ \mathcal{H} \end{matrix} \begin{matrix} \begin{matrix} \mathbb{C} & \mathcal{H} \end{matrix} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{matrix}, \tag{1.11}$$

and $\mathcal{E}_\lambda = \lambda^1 I_{\mathcal{H}_1} \oplus \lambda^2 I_{\mathcal{H}_2}$, then

$$\phi(\lambda) \;=\; A + B\mathcal{E}_\lambda(I_{\mathcal{H}} - D\mathcal{E}_\lambda)^{-1}C. \tag{1.12}$$

1.1. Proofs.

DEFINITION 1.13. A kernel k on $\mathbb{D}^2$ is called *admissible* if

$$(1 - \zeta^1\bar{\lambda}^1)\, k(\zeta,\lambda) \;\geq\; 0 \tag{1.14}$$

and

$$(1 - \zeta^2\bar{\lambda}^2)\, k(\zeta,\lambda) \;\geq\; 0. \tag{1.15}$$

If k is an admissible kernel, then the operators T_1 and T_2 defined by

$$T_r \,:\, k_\lambda \;\mapsto\; \bar{\lambda}^r k_\lambda, \qquad r = 1, 2$$

are a pair of commuting contractions on $\mathcal{H}(k)$, the Hilbert function space on the bidisk for which k is the reproducing kernel. The adjoints T_1^* and T_2^* are the operators of multiplication by the coordinate functions, and (1.14) and (1.15) are just the statements that $I - T_1^*T_1$ and $I - T_2^*T_2$ are positive — *i.e.* that T_1 and T_2 are contractions.

Suppose g is a self-adjoint function on $\mathbb{D}^2 \times \mathbb{D}^2$ that has the property that its Schur product with every admissible kernel is positive semi-definite (*i.e.*

$$\sum \bar{c}_i c_j \, g(\lambda_i,\lambda_j)\, k(\lambda_i,\lambda_j) \;\geq\; 0$$

for every admissible kernel k and every finite set of points $\{\lambda_i\}$ and scalars $\{c_i\}$). One way this could happen is if there were a representation

$$g(\zeta,\lambda) \;=\; (1 - \zeta^1\bar{\lambda}^1)\Gamma(\zeta,\lambda) \,+\, (1 - \zeta^2\bar{\lambda}^2)\Delta(\zeta,\lambda), \tag{1.16}$$

for some kernels Γ and Δ. Indeed, by the Schur Product Theorem, the Schur product of any admissible kernel with the right-hand side of (1.16) is automatically positive. The following structure theorem says that g having the form of (1.16) is not only sufficient, but also necessary.

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THEOREM 1.17. *Let $g : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ be self-adjoint (i.e. $g(\lambda,\zeta) = \overline{g(\zeta,\lambda)}$). Suppose that*

$$g \cdot k \,:\, (\zeta,\lambda) \;\mapsto g(\zeta,\lambda)k(\zeta,\lambda)$$

is positive semi-definite for every admissible kernel k. Then there are positive semi-definite functions Γ and Δ such that

$$g(\zeta,\lambda) \;=\; (1-\zeta^1\bar\lambda^1)\Gamma(\zeta,\lambda) \,+\, (1-\zeta^2\bar\lambda^2)\Delta(\zeta,\lambda). \tag{1.18}$$

For a proof, see [**5**]. (The idea of the proof is to argue by contradiction. If g does not have the desired form, then by the Hahn-Banach theorem one can separate everything on the right-hand-side of (1.18) from g by a linear functional. One uses this to produce an admissible kernel whose Schur product with g is not positive).

PROOFS OF THEOREMS 1.8 AND 1.10.

(Necessity) Suppose ϕ is in the closed unit ball of $H^\infty(\mathbb{D}^2)$, which we shall write as $H^\infty_1(\mathbb{D}^2)$. For simplicity, we shall assume furthermore that ϕ is continuous on the closed bidisk, so it lies in the bidisk algebra $A(\mathbb{D}^2)$. (This restriction can be dropped by using a limiting argument, which we shall omit). Let k be any admissible kernel. The fact that k is admissible means that the operators T_1 and T_2, defined by

$$T_r \,:\, k_\lambda \;\mapsto\; \bar\lambda^r k_\lambda, \qquad r = 1,2,$$

are commuting contractions on $\mathcal{H}_k$. We want to use Andô's inequality [**8**] to conclude that $\phi(T_1,T_2)$ is a contraction. Andô's inequality, which will be discussed in detail in Section 3, says that if T_1 and T_2 are commuting contractions, and ϕ is in the bidisk algebra $A(\mathbb{D}^2)$, the uniform closure of the polynomials in the supremum norm on the bidisk, then

$$\|\phi(T_1,T_2)\| \;\le\; \|\phi\|_{\mathbb{D}^2}.$$

We must make a technical adjustment: we must work not with ϕ but with $\check\phi$ (we define $\check\phi$ by $\check\phi(\lambda^1,\lambda^2) := \phi(\bar\lambda^1,\bar\lambda^2)^*$, and so it is also in the closed unit ball of $A(\mathbb{D}^2)$).

Then, by Andô's inequality, $\check\phi(T_1,T_2)$ is a contraction, so for every finite set of points $\{\lambda_i\}$ in $\mathbb{D}^2$ and scalars c_i, we have

$$\begin{aligned} 0 \quad &\le \quad \Big\langle\, (I - \check\phi(T_1,T_2)\check\phi(T_1,T_2)^*) \sum_j c_j k_{\lambda_j}, \sum_i c_i k_{\lambda_i} \,\Big\rangle \\ &= \quad \sum_{i,j} \bar c_i c_j (1 - \phi(\lambda_i)\overline{\phi(\lambda_j)})\langle k_{\lambda_j}, k_{\lambda_i}\rangle. \end{aligned}$$

Therefore $1 - \phi(\zeta)\overline{\phi(\lambda)}$ satisfies the hypotheses in Theorem 1.17, and so there is a representation

$$1 - \phi(\zeta)\overline{\phi(\lambda)} \;=\; (1-\zeta^1\bar\lambda^1)\Gamma(\zeta,\lambda) \,+\, (1-\zeta^2\bar\lambda^2)\Delta(\zeta,\lambda) \tag{1.19}$$

for some kernels Γ and Δ.

These kernels can be represented as

$$\begin{aligned} \Gamma(\zeta,\lambda) \quad &= \quad \langle g_1(\zeta), g_1(\lambda)\rangle_{\mathcal{H}_1} \\ \Delta(\zeta,\lambda) \quad &= \quad \langle g_2(\zeta), g_2(\lambda)\rangle_{\mathcal{H}_2} \end{aligned}$$

for some functions $g_r : \mathbb{D}^2 \to \mathcal{H}_r$ and some auxiliary Hilbert spaces $\mathcal{H}_r$.

Using these representations, (1.19) becomes

$$1 - \phi(\zeta)\overline{\phi(\lambda)} \;=\; (1-\zeta^1\bar\lambda^1)\langle g_1(\zeta), g_1(\lambda)\rangle_{\mathcal{H}_1} \,+\, (1-\zeta^2\bar\lambda^2)\langle g_2(\zeta), g_2(\lambda)\rangle_{\mathcal{H}_2} \tag{1.20}$$

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and so

$$\begin{aligned} 1 + \zeta^1\bar{\lambda}^1\langle g_1(\zeta), g_1(\lambda)\rangle_{\mathcal{H}_1} + \zeta^2\bar{\lambda}^2\langle g_2(\zeta), g_2(\lambda)\rangle_{\mathcal{H}_2} \\ = \phi(\zeta)\overline{\phi(\lambda)} + \langle g_1(\zeta), g_1(\lambda)\rangle_{\mathcal{H}_1} + \langle g_2(\zeta), g_2(\lambda)\rangle_{\mathcal{H}_2}. \end{aligned} \tag{1.21}$$

Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and let $g(\lambda) = g_1(\lambda) \oplus g_2(\lambda)$. Then (1.21) says that if V is defined by

$$V : \begin{pmatrix} 1 \\ \mathcal{E}_\lambda g(\lambda) \end{pmatrix} \mapsto \begin{pmatrix} \phi(\lambda) \\ g(\lambda) \end{pmatrix}, \tag{1.22}$$

then V extends linearly to an isometry on the span of these elements, and, adding an infinite-dimensional summand to $\mathcal{H}$ if necessary, can then be extended to an isometry from $\mathbb{C} \oplus \mathcal{H}$ to $\mathbb{C} \oplus \mathcal{H}$. Writing V as in (1.11) and solving for ϕ in (1.22), we get that

$$\phi(\lambda) = A + B\mathcal{E}_\lambda(I_{\mathcal{H}} - D\mathcal{E}_\lambda)^{-1}C,$$

as desired.

(Sufficiency) Suppose ϕ can be written as in (1.12), which we have shown is equivalent to (1.9). By expanding $(I - D\mathcal{E}_\lambda)^{-1}$ in a Neumann series, it is clear that ϕ can be written as a power series that converges in $\mathbb{D}^2$, so is analytic there.

To prove that $\|\phi\|$ is bounded by 1, we use the fact that V is an isometry to get

$$\begin{aligned} 1 - \phi(\lambda)^*\phi(\lambda) \\ &= I - A^*A - A^*B\mathcal{E}_\lambda(I - D\mathcal{E}_\lambda)^{-1}C - C^*(I - \mathcal{E}_\lambda^*D^*)^{-1}\mathcal{E}_\lambda^*B^*A \\ &\quad - C^*(I - \mathcal{E}_\lambda^*D^*)^{-1}\mathcal{E}_\lambda^*B^*B\mathcal{E}_\lambda(I - D\mathcal{E}_\lambda)^{-1}C \\ &= C^*C + C^*D\mathcal{E}_\lambda(I - D\mathcal{E}_\lambda)^{-1}C + C^*(I - \mathcal{E}_\lambda^*D^*)^{-1}\mathcal{E}_\lambda^*D^*C \\ &\quad - C^*(I - \mathcal{E}_\lambda^*D^*)^{-1}\mathcal{E}_\lambda^*(I - D^*D)\mathcal{E}_\lambda(I - D\mathcal{E}_\lambda)^{-1}C \\ &= C^*(I - \mathcal{E}_\lambda^*D^*)^{-1}\left[(I - \mathcal{E}_\lambda^*D^*)(I - D\mathcal{E}_\lambda) + (I - \mathcal{E}_\lambda^*D^*)D\mathcal{E}_\lambda\right. \\ &\quad \left. + \mathcal{E}_\lambda^*D^*(I - D\mathcal{E}_\lambda) - \mathcal{E}_\lambda^*(I - D^*D)\mathcal{E}_\lambda\right](I - D\mathcal{E}_\lambda)^{-1}C \\ &= C^*(I - \mathcal{E}_\lambda^*D^*)^{-1}\left[I - \mathcal{E}_\lambda^*\mathcal{E}_\lambda\right](I - D\mathcal{E}_\lambda)^{-1}C. \end{aligned} \tag{1.23}$$

The last expression (1.23) is positive when λ is in $\mathbb{D}^2$, so $\|\phi\|$ is bounded by 1 in the bidisk, as desired. □

2. Lecture 2: Interpolation and Interpolating sequences

The Pick problem on the disk is to determine, given N points $\lambda_1, \ldots, \lambda_N$ in $\mathbb{D}$ and N complex numbers $w_1, \ldots, w_N$, whether there exists $\phi \in H_1^\infty(\mathbb{D})$ such that

$$\phi(\lambda_i) = w_i, \qquad i = 1, \ldots, N.$$

G. Pick proved [**19**] that the answer is yes if and only if the N-by-N matrix

$$\left(\frac{1 - w_i\overline{w}_j}{1 - \lambda_i\bar{\lambda}_j}\right) \tag{2.1}$$

is positive semi-definite.

Pick's theorem on the bidisk was proved by J. Agler [**2**].

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THEOREM 2.2. *Given points $\lambda_1, \ldots, \lambda_N$ in $\mathbb{D}^2$ and complex numbers $w_1, \ldots, w_N$, there is a function $\phi \in H^\infty_1(\mathbb{D}^2)$ that maps each λ_i to the corresponding w_i if and only if there are positive semi-definite matrices Γ and Δ such that*

$$1 - w_i \bar{w}_j = (1 - \lambda_i^1 \bar{\lambda}_j^1)\Gamma_{ij} + (1 - \lambda_i^2 \bar{\lambda}_j^2)\Delta_{ij}. \tag{2.3}$$

Theorem 2.2 can be proved by representing the matrices Γ and Δ as Grammians, as in the transition from (1.19) to (1.20), rearranging the equation as in (1.21), and then introducing the lurking isometry V as in (1.22). Writing this V as in (1.6), the function ϕ from (1.7) can be shown to solve the interpolation problem (and also to be a rational inner function).

Given a sequence $\{\lambda_i\}_{i=1}^\infty$ in the polydisk $\mathbb{D}^d$, we say it is interpolating for $H^\infty(\mathbb{D}^d)$ if, for any bounded sequence $\{w_i\}_{i=1}^\infty$, there is a function ϕ in $H^\infty(\mathbb{D}^d)$ satisfying $\phi(\lambda_i) = w_i$. L. Carleson characterized interpolating sequences on $\mathbb{D}$ in [**12**].

Before stating his theorem, let us introduce some definitions. Given any kernel k on $\mathbb{D}^d$, a sequence $\{\lambda_i\}_{i=1}^\infty$ has an associated Grammian G^k, where

$$[G^k]_{ij} = \frac{k(\lambda_i, \lambda_j)}{\sqrt{k(\lambda_i, \lambda_i)\, k(\lambda_j, \lambda_j)}}.$$

We think of G^k as an infinite matrix, representing an operator on ℓ^2 (that is not necessarily bounded). When k is the Szegő kernel on $\mathbb{D}^d$,

$$k^S(\zeta, \lambda) = \frac{1}{(1 - \zeta^1 \bar{\lambda}^1)(1 - \zeta^2 \bar{\lambda}^2) \cdots (1 - \zeta^d \bar{\lambda}^d)}, \tag{2.4}$$

we call the associated Grammian the *Szegő Grammian*. The Szegő kernel is the reproducing kernel for the Hardy space $H^2(\mathbb{D}^d)$.

An analogue on the polydisk of the pseudo-hyperbolic metric is the *Gleason distance*, defined by

$$\rho(\zeta, \lambda) := \sup\{|\phi(\zeta)| : \|\phi\|_{H^\infty(\mathbb{D}^d)} \leq 1, \phi(\lambda) = 0\}.$$

We shall call a sequence $\{\lambda_i\}_{i=1}^\infty$ *weakly separated* if there exists $\varepsilon > 0$ such that, for all $i \neq j$, the Gleason distance $\rho(\lambda_i, \lambda_j) \geq \varepsilon$. We call the sequence *strongly separated* if there exists $\varepsilon > 0$ such that, for all i, there is a function ϕ_i in $H^\infty_1(\mathbb{D})$ such that

$$\phi_i(\lambda_j) = \begin{cases} \varepsilon, & j = i \\ 0, & j \neq i \end{cases}$$

In $\mathbb{D}$, a straightforward argument using Blaschke products shows that a sequence is strongly separated if and only if

$$\prod_{j \neq i} \rho(\lambda_i, \lambda_j) \geq \varepsilon \qquad \forall\ i.$$

We can now state Carleson's theorem. He proved it using function theoretic methods, but later H. Shapiro and A. Shields [**22**] found a Hilbert space approach, which has proved to be more easily generalized, *e.g.* to characterizing interpolating sequences in the multiplier algebra of the Dirichlet space [**18**]. For a unified treatment, see the lovely monograph [**21**] by K. Seip.

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THEOREM 2.5. *On the unit disk, the following are equivalent:*
(1) *There exists* $\varepsilon > 0$ *such that*

$$\prod_{j \neq i} \rho(\lambda_i, \lambda_j) \geq \varepsilon \qquad \forall\ i.$$

(2) *The sequence* $\{\lambda_i\}_{i=1}^{\infty}$ *is an interpolating sequence for* $H^\infty(\mathbb{D})$.
(3) *The sequence* $\{\lambda_i\}_{i=1}^{\infty}$ *is weakly separated and the associated Szegő Grammian is a bounded operator on* ℓ^2.

In 1987 B. Berndtsson, S.-Y. Chang and K.-C. Lin proved the following theorem [**10**]:

THEOREM 2.6. *Let* $d \geq 2$. *Consider the three statements*
(1) *There exists* $\varepsilon > 0$ *such that*

$$\prod_{j \neq i} \rho(\lambda_i, \lambda_j) \geq \varepsilon \qquad \forall i.$$

(2) *The sequence* $\{\lambda_i\}_{i=1}^{\infty}$ *is an interpolating sequence for* $H^\infty(\mathbb{D}^d)$.
(3) *The sequence* $\{\lambda_i\}_{i=1}^{\infty}$ *is weakly separated and the associated Szegő Grammian is a bounded operator on* ℓ^2.
Then (1) *implies* (2) *and* (2) *implies* (3). *Moreover the converses of these implications are false.*

For the following theorem, which was proved in [**4**], let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for ℓ^2. Recall from Definition 1.13 that a kernel k on $\mathbb{D}^2$ is admissible if the function $(1 - \zeta^r \bar{\lambda}^r)k(\zeta, \lambda)$ is positive semidefinite for r equal to 1 and 2.

THEOREM 2.7. *Let* $\{\lambda_i\}_{i=1}^{\infty}$ *be a sequence in* $\mathbb{D}^2$. *The following are equivalent:*
(i) $\{\lambda_i\}_{i=1}^{\infty}$ *is an interpolating sequence for* $H^\infty(\mathbb{D}^2)$.
(ii) *The following two conditions hold.*
(a) *For all admissible kernels* k, *their normalized Grammians are uniformly bounded:*

$$G^k \ \leq \ MI$$

for some positive constant M.
(b) *For all admissible kernels* k, *their normalized Grammians are uniformly bounded below:*

$$NG^k \ \geq \ I$$

for some positive constant N.
(iii) *The sequence* $\{\lambda_i\}_{i=1}^{\infty}$ *is strongly separated and condition* (a) *alone holds.*
(iv) *Condition* (b) *alone holds.*
Moreover, Condition (a) *is equivalent to both* (a′) *and* (a″)*:*
(a′)*: There exists a constant* M *and positive semi-definite infinite matrices* Γ^1 *and* Γ^2 *such that*

$$M\delta_{ij} - 1 \ = \ \Gamma^1_{ij}(1 - \bar{\lambda}^1_i \lambda^1_j) + \Gamma^2_{ij}(1 - \bar{\lambda}^2_i \lambda^2_j).$$

(a″)*: There exists a function* Φ *in* $H^\infty(\mathbb{D}^2, B(\ell^2, \mathbb{C}))$ *of norm at most* $\sqrt{M}$ *such that* $\Phi(\lambda_i)e_i = 1$.
Condition (b) *is equivalent to both* (b′) *and* (b″)*:*
(b′)*: There exists a constant* N *and positive semi-definite infinite matrices* Δ^1 *and* Δ^2 *such that*

$$N - \delta_{ij} \ = \ \Delta^1_{ij}(1 - \bar{\lambda}^1_i \lambda^1_j) + \Delta^2_{ij}(1 - \bar{\lambda}^2_i \lambda^2_j).$$

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(b″): *There exists a function* Ψ *in* $H^\infty(\mathbb{D}^2, B(\mathbb{C}, \ell^2))$ *of norm at most* $\sqrt{N}$ *such that* $\Psi(\lambda_i) = e_i$.

Neither Theorem 2.6 nor 2.7 are fully satisfactory. For example, the following is still an unsolved problem:

QUESTION 2.8. If a sequence on $\mathbb{D}^2$ is strongly separated, is it an interpolating sequence?

3. Lecture 3: Distinguished Varieties and Andô's Inequality

Let $\mathbb{E}$ be the exterior of the closed disk, $\mathbb{C} \setminus \overline{\mathbb{D}}$. We call an algebraic set V a distinguished variety if

$$V \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2.$$

Von Neumann's inequality [**24**] says that if T is a contraction (a Hilbert space operator of norm at most one), then for any polynomial p,

$$\|p(T)\| \leq \|p\|_{\mathbb{D}}.$$

Andô's inequality [**8**] is a two-variable analogue. It says that if $T = (T_1, T_2)$ is a pair of commuting contractions, then

$$\|p(T)\| \leq \|p\|_{\mathbb{D}^2}. \tag{3.1}$$

Both von Neumann's and Andô's inequality extend automatically to functions in the norm-closure of the polynomials, *viz.* the disk and bidisk algebras respectively. Provided one sticks to operators for which the H^∞ functional calculus makes sense, the inequalities also extend to H^∞.

In [**6**] it was shown that if T is a pair of commuting contractive *matrices*, then there is a distinguished variety V so that (3.1) can be sharpened to

$$|p(T)\| \leq \|p\|_{V \cap \mathbb{D}^2}.$$

Distinguished varieties turn out to be intimately connected to function theory on $\mathbb{D}^2$.

3.1. Representing Distinguished Varieties. For positive integers m and n, let

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C}^m \oplus \mathbb{C}^n \to \mathbb{C}^m \oplus \mathbb{C}^n \tag{3.2}$$

be an $(m+n)$-by-$(m+n)$ unitary matrix. Let

$$\Psi(z) = A + zB(I - zD)^{-1}C \tag{3.3}$$

be the m-by-m matrix valued function defined on the unit disk $\mathbb{D}$ by the entries of U. This is called the *transfer function* of U. Because $U^*U = I$, a calculation (essentially the same as (1.23), but with $\mathcal{E}_\lambda$ replaced by λI) yields

$$I - \Psi(z)^*\Psi(z) = (1 - |z|^2)\, C^*(I - \bar{z}D^*)^{-1}(I - zD)^{-1}C, \tag{3.4}$$

so $\Psi(z)$ is a rational matrix-valued function that is unitary on the unit circle and contractive on the unit disk. Such functions are called rational matrix inner functions, and it is well-known that all rational matrix inner functions have the form (3.3) for some unitary matrix decomposed as in (3.2) — see *e.g.* [**5**] for a proof.

Let V be the set

$$V = \{(z, w) \in \mathbb{D}^2 : \det(\Psi(z) - wI) = 0\}. \tag{3.5}$$

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We shall show that V is a distinguished variety, and that every distinguished variety arises this way — Theorem 3.12 below.

LEMMA 3.6. *Let*

$$U' \;=\; \begin{pmatrix} D^* & B^* \\ C^* & A^* \end{pmatrix} \;:\; \mathbb{C}^n \oplus \mathbb{C}^m \;\to\; \mathbb{C}^n \oplus \mathbb{C}^m,$$

let

$$\Psi'(z) \;=\; D^* + zB^*(I - zA^*)^{-1}C^*,$$

and let

$$V' \;=\; \{(z,w) \in \mathbb{D}^2 \;:\; \det(\Psi'(w) - zI) = 0\}.$$

Then $V = V'$.

PROOF. The point $(z,w) \in \mathbb{D}^2$ is in V iff there is a non-zero vector v_1 in $\mathbb{C}^m$ such that

$$\big[A + zB(1 - zD)^{-1}C\big]\, v_1 \;=\; wv_1. \tag{3.7}$$

Claim: (3.7) holds if and only if there is a non-zero vector v_2 in $\mathbb{C}^n$ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} v_1 \\ z\, v_2 \end{pmatrix} \;=\; \begin{pmatrix} w\, v_1 \\ v_2 \end{pmatrix}. \tag{3.8}$$

PROOF OF CLAIM: If (3.8) holds, then solving gives (3.7). Conversely, if (3.7) holds, define

$$v_2 \;=\; (I - zD)^{-1}Cv_1.$$

Then (3.8) holds. Moreover, if v_2 were 0, then v_1 would be in the kernel of C and be a w-eigenvector of A. As $A^*A + C^*C = I$, this would force $|w| = 1$, contradicting the fact that $(z,w) \in \mathbb{D}^2$. ◁

Given the claim, the point (z,w) is in V' iff there are non-zero vectors v_1 and v_2 such that

$$\begin{pmatrix} D^* & B^* \\ C^* & A^* \end{pmatrix} \begin{pmatrix} v_2 \\ w\, v_1 \end{pmatrix} \;=\; \begin{pmatrix} z\, v_2 \\ v_1 \end{pmatrix}. \tag{3.9}$$

Interchanging coordinates, (3.9) becomes

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} w\, v_1 \\ v_2 \end{pmatrix} \;=\; \begin{pmatrix} v_1 \\ z\, v_2 \end{pmatrix}. \tag{3.10}$$

Clearly, (3.8) and (3.10) are equivalent. □

Note that if C has a non-trivial kernel $\mathcal{N}$, then (3.4) shows that $\Psi(z)$ is isometric on $\mathcal{N}$ for all z, so by the maximum principle is equal to a constant isometry with initial space $\mathcal{N}$. If C has a trivial kernel, we say Ψ is *pure*. Every rational inner function decomposes into the direct sum of a pure rational inner function and a unitary matrix — see *e.g.* [**23**]. Since $A^*A + C^*C = I$, we see that C has no kernel iff $\|A\| < 1$. Since $AA^* + BB^* = I$, this in turn is equivalent to B^* having no kernel. Therefore Ψ is pure iff Ψ' is.

Let V be a distinguished variety. We say a function f is *holomorphic on V* if, for every point of V, there is an open ball B in $\mathbb{C}^2$ containing the point, and a holomorphic function ϕ of two variables on B, such that $\phi|_{B\cap V} = f|_{B\cap V}$. We shall use $A(V)$ to denote the Banach algebra of functions that are holomorphic on V and

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continuous on $\overline{V}$. This is a uniform algebra on ∂V, *i.e.* a closed unital subalgebra of $C(\partial V)$ that separates points. The maximal ideal space of $A(V)$ is $\overline{V}$.

If μ is a finite measure on a distinguished variety V, let $H^2(\mu)$ denote the closure in $L^2(\mu)$ of the polynomials. We say a point λ is a bounded point evaluation for $H^2(\mu)$ if evaluation at λ, *a priori* defined only for a dense set of analytic functions, extends continuously to the whole Hilbert space. If λ is a bounded point evaluation, we call the function k_λ that has the property that

$$\langle f, k_\lambda \rangle \;=\; f(\lambda)$$

the evaluation functional at λ.

For the proof of the following lemma, see [**6**].

LEMMA 3.11. *Let V be a distinguished variety. There is a measure μ on ∂V such that every point in V is a bounded point evaluation for $H^2(\mu)$, and such that the span of the evaluation functionals is dense in $H^2(\mu)$.*

THEOREM 3.12. *The set V, defined by (3.5) for some rational matrix inner function Ψ, is a distinguished variety. Moreover, every distinguished variety can be represented in this form.*

PROOF. Suppose V is given by (3.5), and that (z, w) is in $\overline{V}$. Without loss of generality, we can assume that Ψ is pure. Indeed, any unitary summand of Ψ would add sheets to the variety $\det(\Psi(z) - wI) = 0$ of the type $\mathbb{C} \times \{w_0\}$, for some unimodular w_0. These sheets are all disjoint from the open bidisk $\mathbb{D}^2$.

If $|z| < 1$, equation (3.4) then shows that $\Psi(z)$ is a strict contraction, so all its eigenvalues must have modulus less than 1, and so $|w| < 1$ also. To prove that $|w| < 1$ implies $|z| < 1$, just apply the same argument to V'. Therefore V is a distinguished variety.

To prove that all distinguished varieties arise in this way, let V be a distinguished variety. Let μ be the measure from Lemma 3.11, and let $H^2(\mu)$ be the closure of the polynomials in $L^2(\mu)$. The set of bounded point evaluations for $H^2(\mu)$ is precisely V. (It cannot be larger, because $\overline{V}$ is polynomially convex, and Lemma 3.11 ensures that it is not smaller).

Let $T = (T_1, T_2)$ be the pair of operators on $H^2(\mu)$ given by multiplication by the coordinate functions. They are pure commuting isometries[1] because the span of the evaluation functionals is dense. The joint eigenfunctions of their adjoints are the evaluation functionals.

By the Sz.-Nagy-Foiaş model theory [**23**], T_1 can be modelled as M_z, multiplication by the independent variable z on $H^2 \otimes \mathbb{C}^m$, a vector-valued Hardy space on the unit circle. In this model, T_2 can be modelled as M_Ψ, multiplication by $\Psi(z)$ for some pure rational matrix inner function Ψ. A point (z, w) in $\mathbb{D}^2$ is a bounded point evaluation for $H^2(\mu)$ iff $(\bar{z}, \bar{w})$ is a joint eigenvalue for (T_1^*, T_2^*). In terms of the unitarily equivalent Sz.-Nagy-Foiaş model, this is equivalent to $\bar{w}$ being an eigenvalue of $\Psi(z)^*$.

Therefore

$$V = \{(z, w) \in \mathbb{D}^2 \;:\; \det(\Psi(z) - wI) = 0\},$$

as desired. □

[1] A pure isometry S is one that has no unitary summand; this is the same as requiring that $\cap_{i=1}^{\infty} \mathrm{ran}(S^i) = \{0\}$.

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G. Kneses gives a more constructive proof of Theorem 3.12 in [**17**].

If Ψ is the transfer function of a unitary U as in (3.2), and Ψ is pure, we shall say that V is of *rank* (m, n). This means that generically there are m sheets above each z, and n sheets above each w.

3.2. A sharpening of Andô's inequality.

THEOREM 3.13. *Let T_1 and T_2 be commuting contractive matrices, neither of which has eigenvalues of modulus* 1. *Then there is a distinguished variety V such that, for any polynomial p in two variables, the inequality*

$$\|p(T_1, T_2)\| \ \leq \ \|p\|_V \tag{3.14}$$

holds.

PROOF. Let the dimension of the space on which the matrices act be N.

(i) First, let us assume that each T_r has N linearly independent unit eigenvectors, $\{v_j\}_{j=1}^N$. So we have

$$T_r v_j \ = \ \lambda_j^r v_j, \qquad r = 1, 2 \quad 1 \leq j \leq N,$$

for some set of scalars $\{\lambda_j^r\}$. As each T_r is a contraction, we have $I - T_r^* T_r$ is positive semidefinite, so

$$\langle (I - T_r^* T_r) v_j, v_i \rangle \ = \ (1 - \overline{\lambda_i^r} \lambda_j^r) \langle v_j, v_i \rangle \ \geq \ 0. \tag{3.15}$$

As the matrix in (3.15) is positive semidefinite, it can be represented as the Grammian of vectors u_j^r, which can be chosen to lie in a Hilbert space of dimension d_r equal to the defect of T_r (the defect of T_r is the rank of $I - T_r^* T_r$). So we have

$$(1 - \overline{\lambda_i^1} \lambda_j^1) \langle v_j, v_i \rangle \ = \ \langle u_j^1, u_i^1 \rangle \tag{3.16}$$

$$(1 - \overline{\lambda_i^2} \lambda_j^2) \langle v_j, v_i \rangle \ = \ \langle u_j^2, u_i^2 \rangle. \tag{3.17}$$

Multiplying the first equation by $(1 - \overline{\lambda_i^2} \lambda_j^2)$ and the second equation by $(1 - \overline{\lambda_i^1} \lambda_j^1)$, we see that they are equal. Therefore

$$(1 - \overline{\lambda_i^1} \lambda_j^1) \langle u_j^2, u_i^2 \rangle \ = \ (1 - \overline{\lambda_i^2} \lambda_j^2) \langle u_j^1, u_i^1 \rangle. \tag{3.18}$$

Reordering equation (3.18), we get

$$\langle u_j^1, u_i^1 \rangle \ + \ \overline{\lambda_i^1} \lambda_j^1 \langle u_j^2, u_i^2 \rangle \ = \ \langle u_j^2, u_i^2 \rangle \ + \ \overline{\lambda_i^2} \lambda_j^2 \langle u_j^1, u_i^1 \rangle. \tag{3.19}$$

Equation 3.19 says that there is some unitary matrix

$$U \ = \ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ : \ \mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2} \to \mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2} \tag{3.20}$$

such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u_j^1 \\ \lambda_j^1 u_j^2 \end{pmatrix} \ = \ \begin{pmatrix} \lambda_j^2 u_j^1 \\ u_j^2 \end{pmatrix}. \tag{3.21}$$

If the linear span of the vectors $u_j^1 \oplus \lambda_j^1 u_j^2$ is not all of $\mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2}$, then U will not be unique. In this event, we just choose one such U. Define the $d_1 \times d_1$ matrix-valued analytic function Ψ by

$$\Psi(z) \ = \ A + zB(1 - zD)^{-1} C. \tag{3.22}$$

For any function Θ of two variables, scalar or matrix-valued, define

$$\Theta^{\cup}(Z, W) \ := \ [\Theta(Z^*, W^*)]^* .$$

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Let $\Phi = \Psi^\cup$, so

$$\Phi(z) \;=\; A^* + zC^*(1 - zD^*)^{-1}B^*.$$

Equation 3.21 implies that

$$\Psi(\lambda_j^1)u_j^1 \;=\; \left[\Phi(\overline{\lambda_j^1})\right]^* \; u_j^1 \;=\; \lambda_j^2 u_j^1. \tag{3.23}$$

Let s be the Szegő kernel in the Hardy space H^2 of the unit disk (which we called k^S in (1.3)), so

$$s_\lambda(z) \;=\; \frac{1}{1 - \overline{\lambda} z}. \tag{3.24}$$

Let k_j be the vector in $H^2 \otimes \mathbb{C}^{d_1}$ given by

$$k_j \;:=\; s_{\overline{\lambda_j^1}} \otimes u_j^1.$$

Consider the pair of isometries (M_z, M_Φ) on $H^2 \otimes \mathbb{C}^{d_1}$, where M_z is multiplication by the coordinate function (times the identity matrix on $\mathbb{C}^{d_1}$) and M_Φ is multiplication by the matrix function Φ. Then

$$\begin{aligned} M_z^* \;&:\; k_j \mapsto \lambda_j^1 k_j \\ M_\Phi^* \;&:\; k_j \mapsto \lambda_j^2 k_j. \end{aligned}$$

Therefore the map that sends each v_j to k_j gives a unitary equivalence between (T_1, T_2) and the pair (M_z^*, M_Φ^*) restricted to the span of the vectors $\{k_j\}_{j=1}^N$. Therefore the pair (M_z^*, M_Φ^*), acting on the full space $H^2 \otimes \mathbb{C}^{d_1}$, is a co-isometric extension of (T_1, T_2).

Let p be any polynomial (scalar or matrix valued) in two variables. We have

$$\begin{aligned} \|p(T_1, T_2)\| \;&=\; \|p(M_z^*, M_\Phi^*)|_{\vee\{k_j\}}\| \\ &\le\; \|p(M_z^*, M_\Phi^*)\|_{H^2 \otimes \mathbb{C}^{d_1}} \\ &=\; \|p^\cup(M_z, M_\Phi)\|_{H^2 \otimes \mathbb{C}^{d_1}} \\ &\le\; \|p^\cup(M_z, M_\Phi)\|_{L^2 \otimes \mathbb{C}^{d_1}} \\ &=\; \|p^\cup\|_{\partial V^\cup} \end{aligned} \tag{3.25}$$

where $V^\cup$ and V are the sets

$$\begin{aligned} V^\cup \;&=\; \{(z,w) \in \mathbb{D}^2 \;:\; \det(\Phi(z) - wI) = 0\} \\ V \;&=\; \{(z,w) \in \mathbb{D}^2 \;:\; \det(\Psi(z) - wI) = 0\}. \end{aligned} \tag{3.26}$$

Equality (3.25) follows from the observation that

$$\|p^\cup(M_z, M_\Phi)\|_{L^2 \otimes \mathbb{C}^{d_1}} \;=\; \sup_\theta \|p^\cup(e^{i\theta} I, \Phi(e^{i\theta}))\|, \tag{3.27}$$

where the norm on the right is the operator norm on the $d_1 \times d_1$ matrices. Equation (3.4) shows that, except possibly for the finite set $\sigma(D) \cap \mathbb{T}$, the matrix $\Phi(e^{i\theta})$ is unitary, and so the norm of any polynomial applied to $\Phi(e^{i\theta})$ is just the maximum value of the norm of the polynomial on the spectrum of $\Phi(e^{i\theta})$. By continuity, we obtain (3.25). Taking complex conjugates, (3.25) gives

$$\|p(T_1, T_2)\| \;\le\; \|p\|_V,$$

the desired inequality.

By Theorem 3.12, we see that V and $V^\cup$ are distinguished varieties, and by construction, V contains the points $\{(\lambda_j^1, \lambda_j^2) : 1 \le j \le N\}$.

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(ii) Now, we drop the assumption that $T = (T_1, T_2)$ be diagonizable. J. Holbrook proved that the set of diagonizable commuting matrices is dense in the set of all commuting matrices [**14**]. So we can assume that there is a sequence $T^{(n)} = (T_1^{(n)}, T_2^{(n)})$ of commuting matrices that converges to T in norm and such that each pair satisfies the hypotheses of (i), *i.e.* each $T^{(n)}$ is a pair of commuting contractions that have N linearly independent eigenvectors and no unimodular eigenvalues. Each $T^{(n)}$ has a unitary U_n associated to it as in (3.20). By passing to a subsequence if necessary, we can assume that the defects d_1 and d_2 are constant, and that the matrices U_n converge to a unitary U. The corresponding functions Ψ_n from (3.22) will converge to some function Ψ. Let $q_n(z,w) = \det(\Psi_n(z) - wI)$, and $q(z,w) = \det(\Psi(z) - wI)$. Let V be defined by (3.26) for this Ψ, and V_n be the variety corresponding to Ψ_n. Notice that the degrees of q_n are uniformly bounded.

Claim: V is non-empty.

Indeed, otherwise it would contain no points of the form $(0, w)$ for $w \in \mathbb{D}$. That would mean that $\sigma(A) \subseteq \mathbb{T}$, and so B and C would be zero. That in turn would mean that the submatrices A_n in U_n would have all their eigenvalues tending to $\mathbb{T}$, and hence by (3.21), the eigenvalues of $T_2^{(n)}$ would all tend to $\mathbb{T}$. Therefore T_2 would have a unimodular eigenvalue, contradicting the hypotheses.

Claim: V is a distinguished variety.

This follows from Theorem 3.12.

Claim: Inequality (3.14) holds.

This follows from continuity. Indeed, fix some polynomial p. For every $\varepsilon > 0$, for every $n \geq n(\varepsilon)$, we have

$$\begin{aligned} \|p(T)\| &\leq \varepsilon + \|p(T^{(n)})\| \\ &\leq \varepsilon + \|p\|_{V_n}. \end{aligned}$$

We wish to show that

$$\lim_{n\to\infty} \|p\|_{V_n} \;\leq\; \|p\|_V.$$

Suppose not. Then there is some sequence (z_n, w_n) in V_n such that

$$|p(z_n, w_n)| \;\geq\; \|p\|_V + \varepsilon \tag{3.28}$$

for some $\varepsilon > 0$. Moreover, we can assume that (z_n, w_n) converges to some point (z_0, w_0) in $\overline{\mathbb{D}^2}$. The point (z_0, w_0) is in the zero set of q, so if it were in $\mathbb{D}^2$, then it would be in V. Otherwise, (z_0, w_0) must be in $\mathbb{T}^2$. To ensure that (z_0, w_0) is in $\overline{V}$, we must rule out the possibility that some sheet of the zero set of q just grazes the boundary of $\mathbb{D}^2$ without ever coming inside.

But this cannot happen. For every z in $\mathbb{D}$, there are d_1 roots of $\det(\Psi(z) - wI) = 0$, and *all* of these occur in $\mathbb{D}$. So as z tends to z_0 from inside $\mathbb{D}$, one of the d_1 branches of w must tend to w_0 from inside the disk too. Therefore (z_0, w_0) is in the closure of V, and (3.28) cannot happen. □

Remark 1. Once one knows Andô's inequality for matrices, then it follows for all commuting contractions by approximating them by matrices — see [**13**] for an explicit construction. Of course, the set V must be replaced by the limit points of the sets that occur at each stage of the approximation, and in general this may be the whole bidisk.

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Remark 2. In the proof, we actually constructed a co-isometric extension of T that is localized to V, and a unitary dilation of T with spectrum contained in ∂V.

4. Lecture 4: Angular derivatives

The following theorem, called the Julia-Carathéodory theorem, was originally proved by G. Julia [**16**] and C. Carathéodory [**11**].

THEOREM 4.1. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic. Let τ be a point on the unit circle $\mathbb{T}$. The following conditions are equivalent:*

(A) *there exists a sequence $\{\lambda_n\}$ in $\mathbb{D}$ tending to τ such that*

$$\frac{1 - |\phi(\lambda_n)|}{1 - |\lambda_n|}$$

is bounded;

(B) *for every sequence $\{\lambda_n\}$ tending to τ nontangentially,* (4.1) *is bounded;*

(C) *there exist $\omega \in \mathbb{T}$ and $\eta \in \mathbb{C}$ such that*

$$\lim_{\lambda \stackrel{\text{nt}}{\to} \tau} \frac{|\phi(\lambda) - \omega - \eta(\lambda - \tau)|}{|\lambda - \tau|} = 0; \tag{4.2}$$

(D) *there exist $\omega \in \mathbb{T}$ and $\eta \in \mathbb{C}$ such that $\phi(\lambda) \to \omega$ and $\phi'(\lambda) \to \eta$ as $\lambda \to \tau$ nontangentially.*

In two variables, there are natural analogues of conditions (A) - (D). K. Wlodarcczyk [**25**], F. Jafari [**15**] and M. Abate [**1**] obtained generalizations of Theorem 4.1, showing that (A) implies (B) (this is Theorem 4.7 below) and (B) does not imply (C). In [**7**], it was shown that on the bidisk (C) and (D) are equivalent (where derivatives are replaced by gradients, and in the numerator of (4.2) η becomes a 2-vector whose scalar product is taken with the 2-vector $\lambda - \tau$).

4.1. Non-tangential Approach. If $\{\lambda_n\}$ is a sequence in $\mathbb{D}$ and $\tau \in \mathbb{T}$, we say that λ_n *approaches τ nontangentially* if λ_n tends to τ and there exists a constant c such that, for all n,

$$|\tau - \lambda_n| \leq c(1 - |\lambda_n|).$$

We shall make use of a similar notion for the bidisk: if $\{\lambda_n\}$ is a sequence in $\mathbb{D}^2$ and $\tau \in \mathbb{T}^2$, we say that λ_n *approaches τ nontangentially* if λ_n tends to τ and there exists a constant c such that, for all n,

$$||\tau - \lambda_n|| \leq c(1 - ||\lambda_n||). \tag{4.3}$$

We write $\lambda_n \stackrel{\text{nt}}{\to} \tau$. Here and throughout the section $||\cdot||$ on $\mathbb{C}^2$ denotes the ℓ^∞ norm:

$$||\lambda|| = \max\{|\lambda^1|, |\lambda^2|\}.$$

We say that a set S in $\mathbb{D}^2$ approaches a point τ on the torus non-tangentially if τ is in the closure of S and there exists a constant c such that, for all $\lambda \in S$,

$$||\tau - \lambda|| \leq c(1 - ||\lambda||).$$

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4.2. Results for functions on $\mathbb{D}^2$.

DEFINITION 4.4. Let $\phi \in H_1^\infty(\mathbb{D}^2)$ and let $\tau \in \mathbb{T}^2$. We say that τ is a *B-point* for ϕ if there exists a sequence $\{\lambda_n\}$ in $\mathbb{D}^2$ such that

$$\lambda_n \to \tau \text{ and} \tag{4.5}$$

$$\frac{1-|\phi(\lambda_n)|}{1-||\lambda_n||} \text{ is bounded.} \tag{4.6}$$

THEOREM 4.7. *Let ϕ be in $H_1^\infty(\mathbb{D}^2)$. The following are equivalent:*

(A) *the point τ in $\mathbb{T}^2$ is a B-point for ϕ;*
(B) *for every sequence $\{\lambda_n\}$ in $\mathbb{D}^2$ that converges* nt *to τ the statement* (4.6) holds.

When (A) *and* (B) *are satisfied there exists $\omega \in \mathbb{T}$ such that $\phi(\lambda) \to \omega$ as $\lambda_n \overset{\text{nt}}{\to} \tau$.*

There are various ways in which ϕ can have a form of one-sided differentiability at a boundary point. One is for the directional derivative of ϕ at τ in the direction $-\tau\delta$,

$$D_{-\tau\delta}\phi(\tau) = \lim_{t\to 0+} \frac{\phi(\tau - t\tau\delta) - \phi(\tau)}{t} \tag{4.8}$$

to exist whenever δ^1 and δ^2 are in the right half-plane $\mathbb{H}$ (for then $\tau(1-t\delta) \in \mathbb{D}^2$ for small $t > 0$ and the right-hand side of (4.8) makes sense).

Consider the function

$$\psi(\lambda) = \frac{\frac{1}{2}\lambda^1 + \frac{1}{2}\lambda^2 - \lambda^1\lambda^2}{1 - \frac{1}{2}\lambda^1 - \frac{1}{2}\lambda^2}. \tag{4.9}$$

The point $\tau = (1,1)$ is a B-point for ψ, and the nontangential limit there is 1. For every $\delta \in \mathcal{H}$, the directional derivative $D_{-\delta}\psi(1,1)$ exists and

$$D_{-\delta}\psi(1,1) = -\frac{2\,\delta^1\delta^2}{\delta^1 + \delta^2}. \tag{4.10}$$

Notice that the right-hand side of (4.10) is not linear in δ, but is analytic. For a function holomorphic at τ the directional derivative is of course linear in the direction, and so ψ is not regular at $(1,1)$.

(4.10) is typical of behavior at a B-point. In particular, we have:

THEOREM 4.11. *Let τ be a B-point of $\phi \in H_1^\infty(\mathbb{D}^2)$. For any $\delta \in \mathbb{H}^2$ the directional derivative $D_{-\tau\delta}\phi(\tau)$ exists and is an analytic function of δ.*

We say that ϕ *has a holomorphic differential on S at τ* if $S \subset \mathbb{D}^2$, the closure of S contains τ and there exist $\omega, \eta^1, \eta^2 \in \mathbb{C}$ such that, for $\lambda \in S$,

$$\phi(\lambda) = \omega + \eta^1(\lambda^1 - \tau^1) + \eta^2(\lambda^2 - \tau^2) + e(\lambda) \tag{4.12}$$

where

$$\lim_{\lambda\to\tau,\ \lambda\in S} \frac{e(\lambda)}{||\lambda - \tau||} = 0.$$

We say that $\tau \in \mathbb{T}^2$ is a *C-point for ϕ* if, for every set S that approaches τ nontangentially, ϕ has a holomorphic differential on S and ω in (4.12) is unimodular.

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It is clear that, when τ is a C-point for ϕ, the quantities ω, η^1, η^2 in equation (4.12) are the same for every nontangential approach region S, and so we may define the *angular gradient* $\nabla\phi(\tau)$ of ϕ at τ to be the vector $\begin{pmatrix}\eta^1 & \eta^2\end{pmatrix}^t$.

If τ is a C-point of ϕ then the directional derivative $D_{-\tau\delta}\phi(\tau)$ exists for $\delta \in \mathcal{H}$ and

$$D_{-\tau\delta}\phi(\tau) = \delta \cdot \nabla\phi(\tau).$$

Every C-point is a B-point, and in one variable Theorem 4.1 states that the two notions are equivalent. However, the function ψ of equation (4.9) shows that, for functions of two variables, *not every B-point is a C-point*: the relation (4.12) fails to hold for $\phi = \psi$ and $\tau = (1,1)$. Nonetheless, we still have equivalence of the two-variable analogues of conditions (C) and (D) from Theorem 4.1:

THEOREM 4.13. *Let $\tau \in \mathbb{T}^2$ be a C-point for $\phi \in H_1^\infty(\mathbb{D}^2)$. Then*

$$\lim_{\lambda \overset{\text{nt}}{\to} \tau} \nabla\phi(\lambda) \;=\; \nabla\phi(\tau).$$

Points at which ϕ is regular are of course C-points, and the assertion of the theorem is trivial for such C-points, but there are examples of functions in $H_1^\infty(\mathbb{D}^2)$ that have singular C-points. One example is the rational inner function

$$\phi(\lambda) = \frac{-4\lambda^1(\lambda^2)^2 + (\lambda^2)^2 + 3\lambda^1\lambda^2 - \lambda^1 + \lambda^2}{(\lambda^2)^2 - \lambda^1\lambda^2 - \lambda^1 - 3\lambda^2 + 4},$$

which has a C-point at $(1,1)$, despite being singular there (ϕ cannot be extended continuously to $\mathbb{D}^2 \cup \{(1,1)\}$).

Proofs of all the results in this section can be found in [**7**]. The proofs rely very heavily on modelling functions as in (1.20).

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