Problems on Dirichlet Series

John E. McCarthy *
St. Louis, Missouri 63130

July 21 2003

$s := \sigma + it$

$\Omega_\rho := \{s : \Re(s) > \rho\}$.

If $f(s) \sim \sum a_n n^{-s}$, then

$$\begin{aligned}
\sigma_a(f) &:= \inf\{\Re(s) : \sum a_n n^{-s} \text{ converges absolutely}\} \\
\sigma_b(f) &:= \inf\{\rho : \sum a_n n^{-s} \text{ converges to a bounded function in } \Omega_\rho\} \\
\sigma_u(f) &:= \inf\{\rho : \sum a_n n^{-s} \text{ converges uniformly in } \Omega_\rho\} \\
\sigma_c(f) &:= \inf\{\Re(s) : \sum a_n n^{-s} \text{ converges}\}.
\end{aligned}$$

Have [1]: $\sigma_c \leq \sigma_b = \sigma_u \leq \sigma_a \leq \sigma_c + 1$.

Let $\mathcal{D}$ denote the set of functions that can be represented in some half-plane by a Dirichlet series. Let $\mathcal{P}$ denote the set of Dirichlet series with a finite number of terms, and $\mathcal{P}_0$ those with no constant term.

For $\alpha < 0$, define $\mathcal{H}^p_\alpha$ to be the closure of $\mathcal{P}_0$ in the norm

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_0^\infty d\sigma \ \sigma^{-1-\alpha} |f(s+c)|^p;$$

*Partially supported by National Science Foundation Grant DMS 0070639

for $\alpha = 0$ define $\mathcal{H}_0^p$ to be the closure of $\mathcal{P}$ in the norm

$$\lim_{c\to 0^+} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt |f(c+it)|^p. \tag{1}$$

Define $H_\infty^p(\Omega_{\frac{1}{2}})$ to be the holomorphic functions g in $\Omega_{\frac{1}{2}}$ for which

$$\sup_{y\in\mathbb{R}} \sup_{\sigma>\frac{1}{2}} \int_y^{y+1} |g(\sigma+it)|^p$$

is finite.

Theorem 1 *(Carlson [2]). Let $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ be a function in $\mathcal{D}$ that has $\sigma_b = 0$. Then*

$$\sum_{n=1}^{\infty} |a_n|^2 \;=\; \lim_{c\to 0^+} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt |f(c+it)|^2. \tag{2}$$

Problem 1 Prove Theorem 1 with the hypothesis $\sigma_b = 0$ replaced by

(i) $\sigma_c(f) < 0$.

(ii) $\sum_{n=1}^\infty |a_n|^2 < \infty$.

(iii) $\log|f(s)| = O(\log|t|)$.

Problem 2 Develop a "BMOA" type theory so that

(i) every function in "BMOA" with $\sigma_c < 0$ has the integrals (1) finite for every $p < \infty$.

(ii) The alternating zeta function $\zeta_1(s) := \sum_{n=1}^\infty (-1)^n n^{-s}$ is in BMOA. (Every even power of ζ_1 is in $\mathcal{H}_0^2$ [4]).

Problem 3 Does $\mathcal{H}_0^p$ embed in $H_\infty^p(\Omega_{\frac{1}{2}})$ for every $p < \infty$? (It does if p is an even integer [3]).

Problem 4 How can the zero sets for $\mathcal{H}^p_\alpha$ and $\mathrm{Mult}(\mathcal{H}^p_\alpha)$ be classified?

Problem 5 What are the interpolating sequences for $\mathcal{H}^2_0$? What are the interpolating sequences that lie on the real axis?

Problem 6 Is $\mathcal{H}^1_0 = \mathcal{H}^2_0 \otimes \mathcal{H}^2_0$, *i.e.* is every function in $\mathcal{H}^1_0$ writable as a sum of products of functions in $\mathcal{H}^2_0$ with norm control?

References

[1] H. Bohr. Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen $\Sigma a_n/n^s$. *Nachr. Akad. Wiss. Göttingen math.-Phys. Kl.*, pages 441–448, 1913.

[2] F. Carlson. Contributions á la théorie des séries de Dirichlet, Note I. *Ark. Mat.*, 16(18):1–19, 1922.

[3] H. Hedenmalm, P. Lindqvist, and K. Seip. A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0,1)$. *Duke Math. J.*, 86:1–37, 1997.

[4] H. Helson. Compact groups and Dirichlet series. *Ark. Mat.*, 8:139–143, 1969.